1 Rate of Change

When we talk about the rate of change of a function, $f$, we are referring to the rate at which the dependent variable changes as the independent variable increases. If $f$ is a linear function, then the rate of change of $f$ is simply the slope of $f$. For example, the function for converting Celsius temperature ($C$) to Fahrenheit temperature ($F$),

$$ F = \frac{9}{5} C + 32, $$

has a rate of change of $9/5 = 1.8$. This means that Fahrenheit temperature increases by 9 degrees for each 5 degree increase in Celsius temperature or, equivalently, that Fahrenheit temperature increases by 1.8 degrees for each 1 degree increase in Celsius temperature. In general, for linear functions,

$$ f(x) = mx + b, $$

we have

$$ \text{rate of change of } f = m = \frac{f(x) - f(x_0)}{x - x_0} $$

and it doesn’t matter what two points, $x_0$ and $x$, we choose to compute the difference quotient.

The way to define rate of change is not as straightforward for nonlinear functions, $f$, because the value of a difference quotient,

$$ f(x) - f(x_0) $$

$$ x - x_0 $$

1
does depend on the points, $x_0$ and $x$, used in the computation. For example, if we take the function $f(x) = x^2$ and we compute a difference quotient for this function using $x_0 = 1$ and $x = 4$, we obtain

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4^2 - 1^2}{4 - 1} = \frac{15}{3} = 5$$

and if we use $x_0 = 1$ and $x = 3$, we obtain

$$\frac{f(3) - f(1)}{3 - 1} = \frac{3^2 - 1^2}{3 - 1} = \frac{8}{2} = 4.$$  

In each of these difference quotient computations, we are computing the slope of a secant line joining two points on the graph of $f(x) = x^2$. This is illustrated in the figures below.
Figure 1: $f(x) = x^2$ and its tangent line at $(1, 1)$

If we hold the point $x_0 = 1$ fixed and choose any other point $x \neq 1$, the difference quotient is

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^2 - 1^2}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1,$$

from which we can see that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

Graphically, the slopes of the secant lines joining the points $(x_0, f(x_0)) = (1, 1)$ and $(x, f(x)) = (x, x^2)$ become closer and closer to 2 as we choose $x$ closer and closer to 1. The line having slope 2 and passing through the point $(1, 1)$ is the tangent line to the graph of $f$ at the point $(1, 1)$. The tangent line is the graph of the linear function that best approximates $f$ at $x_0 = 1$. (See Figure 1.)

Based on the above discussion, it seems to makes sense to say that the rate of change of the function $f(x) = x^2$ at the point $x_0 = 1$ is 2. This does not mean that $x^2$ changes by 2 units each time $x$ increases by 1 unit, but it **does** mean that if $x$ is very close to $x_0 = 1$, then

$$\frac{\text{change in } f(x)}{\text{change in } x} = \frac{x^2 - 1}{x - 1} \approx 2.$$

In general, for any function, $f$, and any point, $x_0$, contained in an open
interval that is contained in the domain of \( f \), we define

\[
\text{rate of change of } f \text{ at } x_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]

Thus, finding the rate of change of \( f \) at \( x_0 \) is equivalent to finding the slope of \( f \) at \( x_0 \) (as we have been doing in the examples and exercises of previous sections). The rate of change of \( f \) at \( x_0 \) (or, equivalently, the slope of \( f \) at \( x_0 \)) is most often referred to as the derivative of \( f \) at \( x_0 \). We summarize these ideas in the following definition:

**Definition 1** If \( f \) is a function and \( x_0 \) is a point in an open interval that is contained in the domain of \( f \), then the **derivative** of \( f \) at \( x_0 \) is denoted by \( f'(x_0) \) and is defined to be

\[
f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]

if this limit exists.

The derivative of \( f \) at \( x_0 \) is also called the **slope** of \( f \) at \( x_0 \) or the **rate of change** of \( f \) at \( x_0 \). The line passing through the point \((x_0, f(x_0))\) and having slope \( f'(x_0) \) is called the **tangent line** to the graph of \( f \) at the point \((x_0, f(x_0))\). The tangent line is the graph of the linear function,

\[
L(x) = f'(x_0)(x - x_0) + f(x_0),
\]

which we regard as being the linear function that best approximates \( f(x) \) for values of \( x \) near \( x_0 \). (That is, \( f(x) \approx L(x) \) for values of \( x \) near \( x_0 \).)

As an important remark, note that \( f'(x_0) \) is defined only if the limit of the difference quotient exists. If this limit does indeed exist, then we say that \( f \) is **differentiable** at \( x_0 \). However, it might be the case that the difference quotient does not have a well-defined limit (which is a single real number). In this case, we say that \( f \) is not differentiable at \( x_0 \) (or that the slope of \( f \) is not defined at \( x_0 \)). Examples of functions that are not differentiable at certain points are encountered in exercises 8 and 9 of the following exercise set.
1.1 Exercises

For each of the functions, $f$, and points, $x_0$, given in exercises 1–7, find $f'(x_0)$. If possible, use algebra to simplify the difference quotient and to arrive at an exact answer. Otherwise, use the numerical approach to evaluate the difference quotient. After you have found $f'(x_0)$, write the equation of the tangent line of $f$ at $(x_0, f(x_0))$ and graph $f$ along with this tangent line.

1. $f(x) = x^2$ at $x_0 = -8$.
2. $f(x) = x^2$ at $x_0 = 0$.
3. $f(x) = x^2 - 4x + 2$ at $x_0 = 1$.
4. $f(x) = x^3 - 12x$ at $x_0 = 0$.
5. $f(x) = \sin x$ at $x_0 = \pi$.
6. $f(x) = (2.5)^x$ at $x_0 = 0$.
7. $f(x) = \sqrt{x}$ at $x_0 = 9$.

In some cases, the derivative of a function, $f$, at a given point, $x_0$, is not defined. Exercises 8 and 9 give two such examples. In each exercise, explain why the derivative of $f$ at $x_0$ is not defined. Include graphs that illustrate your explanations.

8. $f(x) = 4 - |x|$ at $x_0 = 0$.
9. $f(x) = \sqrt{x}$ at $x_0 = 0$.

10. Is the derivative of $f(x) = 4 - |x|$ at $x_0 = 0.3$ defined? If so, then find it. If not, then explain why not. Include a graph to illustrate your explanation.

11. Is the derivative of $f(x) = \sqrt{x}$ at $x_0 = 0.04$ defined? If so, then find it. If not, then explain why not. Include a graph to illustrate your explanation.
2 Rates of Change in Applied Problems

Knowing how to study rates of change of nonlinear functions is essential in solving many kinds of problems. Such problems commonly arise in Physics, Chemistry, Biology, Finance, and just about everywhere. Nonlinear phenomena are all around us and Calculus is the main tool needed to understand these phenomena.

Here, we consider two problems that involve tanks being filled with water. In both problems, the tank is being filled with water at a constant rate and we wish to study the rate at which the depth of the water is changing. The difference is that in the first example the tank is cylindrical in shape which means that the water depth is a linear function of time; whereas, in the second example the tank is cone-shaped and water depth is a nonlinear function of time. Only the second example requires Calculus. Before beginning these examples, we need to recall the formulas for finding the volumes of a cylinder and a cone: A cylinder with radius $r$ and height $h$ has volume $V = \pi r^2 h$ and a cone with base radius $r$ and height $h$ has volume $V = \frac{1}{3}\pi r^2 h$. (See Figures.) In addition, we will use the volume conversion

$$1 \text{ gal} = 0.13368 \text{ ft}^3.$$

![cylinder diagram]
Example 2 Suppose that a cylindrical tank (pictured) of diameter 12 ft and height 18 ft is being filled with water at the rate of 10 gallons per minute. At what rate (in feet per minute) is the depth of the water changing?

Solution: To find the rate of change of the depth of the water (in feet per minute), we must first write the depth of the water (in feet) as a function
of time (in minutes). Hence we define the variables

\[ t = \text{time (in minutes) from when filling first begins} \]
\[ D = \text{depth of the water in the tank (in feet)}. \]

In our process of writing \( D \) as a function of \( t \), we will also need to work with the volume of the water in the tank so we define

\[ V = \text{volume of the water in the tank (in cubic feet)}. \]

Note that \( D \) and \( V \) are both functions of \( t \) because both the depth and volume of water in the tank change as the tank is being filled.

Since the body of water in the tank is always in the shape of a cylinder, we can use the formula for the volume of a cylinder to obtain the relation

\[ V = \pi (6 \text{ ft})^2 (D \text{ ft}) = 36\pi D \text{ ft}^3. \]  

(1)

Since the tank is being filled at the rate of 10 gal/min and 1 gal = 0.13368 ft\(^3\), the tank is being filled at the rate of

\[ \frac{10 \text{ gal}}{1 \text{ mn}} \cdot \frac{0.13368 \text{ ft}^3}{1 \text{ gal}} = 1.3368 \text{ ft}^3 / \text{ mn}. \]

Assuming that the tank is initially empty (at time \( t = 0 \)), the volume of water in the tank at time \( t \text{ mn} \) is

\[ V = \left( 1.3368 \text{ ft}^3 / \text{ mn} \right) \cdot (t \text{ mn}) = 1.3368t \text{ ft}^3. \]  

(2)

Using equations (1) and (2), we obtain

\[ 36\pi D = 1.3368t. \]

If we divide both sides of this equation by \( 36\pi \), we obtain

\[ D = \frac{1.3368}{36\pi} t \]

Since \( D \) is a linear function of \( t \) with slope

\[ \frac{1.3368}{36\pi} \approx 0.0118, \]

we conclude that the water depth is increasing at the rate of 0.0118 ft / mn.
**Example 3** Suppose that a cone–shaped tank (pictured) of base diameter 16 ft and height 18 ft is being filled with water at the rate of 10 gallons per minute. Assuming that the tank is empty when filling first begins, at what rate is the depth of the water changing five minutes after filling begins?

![Diagram of a cone-shaped tank]

**Solution:** As in the previous example, we must first write the depth of the water (in feet) as a function of time (in minutes). Hence, we define the variables

\[
t = \text{time (in minutes) from when filling first begins}
\]
\[
D = \text{depth of the water in the tank (in feet)}
\]

Since the volume of the water in the tank and the radius of the water surface will be involved in our reasoning, we also define

\[
V = \text{volume of water in the tank (in cubic feet)}
\]
\[
r = \text{radius of the water surface (in feet)}
\]

Note that \(D, V, \) and \(r\) are all functions of \(t\) because the depth, volume, and radius of the water surface all change as the tank is being filled.

Since the body of water in the tank is always in the shape of a cone, we can use the formula for the volume of a cone to obtain the relation

\[
V = \frac{1}{3} \pi r^2 D. \tag{3}
\]
Since water is filling the tank at the rate of 1.3368 ft³/min, then, just as in the previous example, we have

\[ V = 1.3368t \]

which gives us

\[ \frac{1}{3} \pi r^2 D = 1.3368t. \]  \hspace{1cm} (4)

We still don’t have quite what we want because equation (4) expresses a relation between the three variables \( D, r, \) and \( t \). We would like to eliminate \( r \) so that we will have a relation between \( D \) and \( t \) only. Using properties of similar triangles (see figure), we obtain

\[ \frac{r}{D} = \frac{8}{18} = \frac{4}{9} \]

which gives us

\[ r = \frac{4}{9} D. \]  \hspace{1cm} (5)

Substituting equation (5) into equation (4), we obtain

\[ \frac{1}{3} \pi \left( \frac{4}{9} D \right)^2 D = 1.3368t \]
which, upon simplifying, gives us

$$\frac{16\pi}{243} D^3 = 1.3368t.$$  

or

$$D^3 = \frac{324.8424}{16\pi} t.$$  

Defining $k$ to be the constant

$$k = \frac{324.8424}{16\pi} \approx 6.4625,$$

we now have

$$D^3 = kt.$$  

Finally, upon taking the cube root of both sides of the above equation (which is the same as raising both sides to the $1/3$ power), we obtain

$$D = (kt)^{\frac{1}{3}}.$$  

The graph of $D$ as a function of $t$ is shown in Figure 2. (Try graphing it on your calculator.)

We want to find the rate of change of $D$ (with respect to $t$) at time $t_0 = 5$ min. Since the formula (6) expresses $D$ as a function of $t$, the difference quotient that we are interested in is

$$\frac{(k \cdot t)^{\frac{1}{3}} - (k \cdot 5)^{\frac{1}{3}}}{t - 5}.$$
and we want to compute the limit of this difference quotient as \( t \) approaches \( t_0 = 5 \). Using the numerical approach (choosing values of \( t \) close to 5), we obtain the values given in the following table

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \frac{(k \cdot t)^{\frac{1}{3}} - (k \cdot 5)^{\frac{1}{3}}}{t - 5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.9</td>
<td>0.21377213</td>
</tr>
<tr>
<td>5.1</td>
<td>0.21094052</td>
</tr>
<tr>
<td>4.98</td>
<td>0.2126243</td>
</tr>
<tr>
<td>5.02</td>
<td>0.2120581</td>
</tr>
<tr>
<td>5.00046</td>
<td>0.2123347826</td>
</tr>
</tbody>
</table>

Based on our numerical calculations, we estimate that

\[
\lim_{t \to 5} \frac{(k \cdot t)^{\frac{1}{3}} - (k \cdot 5)^{\frac{1}{3}}}{t - 5} \approx 0.212.
\]

We conclude that at time \( t_0 = 5 \), the depth of the water in the tank is increasing at a rate of about 0.212 ft/min.

### 2.1 Exercises

The following exercises relate to Examples 2 and 3.

1. In Example 2, the problem to be solved was “At what rate is the depth of the water changing?”; whereas, in Example 3, the problem to be solved was “At what rate is the depth of the water changing five minutes after filling begins?” Why would it not make sense in Example 3, to simply have asked “At what rate is the depth of the water changing?”?

2. In Example 2, the height of the cylinder played no role in determining the rate at which the depth of the water is changing. Why not?

3. Referring to Example 3, find the rate of change of the water depth 3 minutes after filling begins and 8 minutes after filling begins.

4. Suppose that a cylindrical tank of diameter 10 ft and height 32 ft is being filled with water at the rate of 20 gallons per minute. At what rate is the depth of the water changing? Show all details of your work following a procedure similar to the solution in Example 2.
5. Suppose that a cone–shaped tank of base diameter 16 ft and height 32 ft is being filled with water at the rate of 10 gallons per minute. Assuming that the tank is empty when filling first begins, at what rate is the depth of the water changing five minutes after filling begins? Show all details of your work following a procedure similar to the solution in Example 3.

6. Suppose that a cylindrical tank of diameter \( a \) ft and height \( b \) ft is being filled with water at the rate of \( R \) gallons per minute. Assuming that the tank is empty when filling first begins and letting \( t \) denote time (in minutes) and letting \( D \) denote the depth of the water in the tank (in feet), find a formula that expresses \( D \) as a function of \( t \). Also, find the rate at which the depth of the water is changing.

7. Suppose that a cone–shaped tank of base diameter \( a \) ft and height \( b \) ft is being filled with water at the rate of \( R \) gallons per minute. Assuming that the tank is empty when filling first begins and letting \( t \) denote time (in minutes) and letting \( D \) denote the depth of the water in the tank (in feet), find a formula that expresses \( D \) as a function of \( t \). Also, write the limit of a difference quotient that one would need to compute in order to find the rate at which the water depth is changing at some given time \( t_0 \).
3 The Derivative Function

The derivative function of a function, $f$, is the function that tells us the derivative of $f$ at each point, $x_0$, in the domain of $f$. The notation $f'$ is used to denote the derivative function of $f$. That is,

$$f' (x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$ 

Usually, we simply call $f'$ “the derivative of $f$” rather than “the derivative function of $f$”. However, $f'$ actually is a function because it assigns values to each point in the domain of $f$ (at which the derivative of $f$ is defined). The process of finding the derivative of a function is called differentiation.

The procedure for differentiation of a function, $f$, is the same procedure used in finding the derivative of $f$ at a particular point $x_0$, except we do not use specific numerical values for $x_0$. In essence, we do all of the work “in one fell swoop” rather than computing derivatives at individual points.

Example 4 In this example, we find the derivative function of the function $f(x) = x^2$. (Note that we make no reference to any particular point $x_0$. This is because we are going to find the derivative of $f$ at all points $x_0$ and we are going to do it all at once.)

For any given point $x_0$, the derivative of $f$ at $x_0$ is

$$f' (x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}.$$ 

To compute the limit of this difference quotient, we use the same algebraic “trick” that was used in previous examples where we were dealing with the function $f(x) = x^2$ and specific values of $x_0$. In particular, we use the factorization

$$x^2 - x_0^2 = (x - x_0)(x + x_0)$$

to rewrite the difference quotient as

$$\frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0.$$ 

From this simplification of the difference quotient, we obtain

$$f' (x_0) = \lim_{x \to x_0} (x + x_0) = x_0 + x_0 = 2x_0.$$
Our conclusion is that the derivative of \( f \) at any point \( x_0 \) is \( f'(x_0) = 2x_0 \).

Notice the great economy that is achieved in this process. We can use the formula \( f'(x_0) = 2x_0 \) to find the derivative of \( f \) at any point without having to go through all of the work separately at each point. We have done the hard work once and for all. As a reality check, let us use the formula \( f'(x_0) = 2x_0 \) to compute the derivative of \( f \) at a sampling of points:

- The derivative of \( f \) at \( x_0 = 1 \) is \( f'(1) = 2 \cdot 1 = 2 \).
- The derivative of \( f \) at \( x_0 = 2 \) is \( f'(2) = 2 \cdot 2 = 4 \).
- The derivative of \( f \) at \( x_0 = -3 \) is \( f'(-3) = 2 \cdot (-3) = -6 \).
- The derivative of \( f \) at \( x_0 = 0 \) is \( f'(0) = 2 \cdot 0 = 0 \).

The results of the above sample computations appear to be consistent with the graph of \( f \) shown in Figure 3.

As a final remark, we stress that \( f' \) is a function. Instead of writing \( f'(x_0) = 2x_0 \), we can write \( f'(x) = 2x \) (using \( x \) rather than \( x_0 \) to denote the independent variable of the function \( f' \)). The graph of the function \( f' \) is shown in Figure 4. Note how the graphs of \( f \) and \( f' \) compare: The value of \( f' \) at any point \( x \) tells us the rate of change or the slope of the tangent line to the graph of \( f \) at that same point \( x \).
Figure 3: Graph of $f(x) = x^2$.

Figure 4: Graph of $f'(x) = 2x$. 
Example 5 In this example, we find the derivative (function) of the function $f(x) = \sqrt{x}$. The derivative of $f$ (at any point $x_0 > 0$) is given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}.$$  

Using a familiar algebraic trick to simplify the difference quotient, we obtain

$$\frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \cdot \frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}}$$

$$= \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{(x - x_0)(\sqrt{x} + \sqrt{x_0})}$$

$$= \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x_0}}.$$

Since

$$\lim_{x \to x_0} \left( \frac{1}{\sqrt{x} + \sqrt{x_0}} \right) = \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}},$$

we obtain

$$f'(x_0) = \frac{1}{2\sqrt{x_0}}.$$

Doing some sample calculations with this formula, we obtain

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} = 0.5$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} = 0.25$$

$$f'(12) = \frac{1}{2\sqrt{12}} \approx 0.1443375673.$$  

The graphs of the functions $f(x) = \sqrt{x}$ and $f'(x) = 1/(2\sqrt{x})$ are shown in Figures 5 and 6.
Figure 5: Graph of $f(x) = \sqrt{x}$.

Figure 6: Graph of $f'(x) = 1/(2\sqrt{x})$. 
3.1 Exercises

In exercises 1–8, find the derivative functions, \( f' \), of the given functions \( f \). Also, graph both \( f \) and \( f'' \).

1. \( f(x) = -4x + 6 \)
2. \( f(x) = x^2 - 4x + 2 \).
3. \( f(x) = -6x^2 + 3 \)
4. \( f(x) = x^3 \)
5. \( f(x) = x^3 - 12x \)
6. \( f(x) = \frac{1}{x} \)
7. \( f(x) = \frac{1}{x^2 + 1} \).
8. \( f(x) = 4 - |x| \). (HINT: The derivative is a piecewise-defined function and \( f'(0) \) is not defined. Why is \( f'(0) \) not defined?)

In exercises 9 and 10, you will find that you cannot use algebra to find exact formulas for the derivatives of the given functions, \( f \). For each given function, \( f \), graph \( f \) and then do your best to draw the graph of \( f' \) (by hand). A suggested way to go about this is to look at the graph of \( f \) and determine intervals where \( f'(x) > 0 \), intervals where \( f'(x) < 0 \), and points where \( f'(x) = 0 \).

9. \( f(x) = \sin x \)
10. \( f(x) = 2^x \)