1 Exercise Set 2

1. We want to establish that the derivative of \( f(x) = x^3 \) is \( f'(x) = 3x^2 \).

To do this, first let \( x_0 \) be a fixed but arbitrary real number.

Using formulation (1) of the definition of the derivative, we have the difference quotient

\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^3 - x_0^3}{x - x_0} = \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = x^2 + xx_0 + x_0^2.
\]

Thus,

\[ f'(x_0) = \lim_{x \to x_0} (x^2 + xx_0 + x_0^2) = x_0^2 + x_0x_0 + x_0^2 = 3x_0^2. \]

Using formulation (2) of the definition of the derivative, we have the difference quotient

\[
\frac{f(x_0 + h) - f(x_0)}{h} = \frac{(x_0 + h)^3 - x_0^3}{h} = \frac{x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 - x_0^3}{h} = \frac{3x_0^2h + 3x_0h^2 + h^3}{h} = 3x_0^2 + 3x_0h + h^2.
\]

Thus,

\[ f'(x_0) = \lim_{h \to 0} (3x_0^2 + 3x_0h + h^2) = 3x_0^2 + 3x_0 \cdot 0 + (0)^2 = 3x_0^2. \]

3. We want to establish that the derivative of \( f(x) = \frac{1}{x} \) is \( f'(x) = -\frac{1}{x^2} \).

To do this, first let \( x_0 \) be a fixed but arbitrary real number (with \( x_0 \neq 0 \)).

Using formulation (1) of the definition of the derivative, we have the difference quotient

\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} = \frac{1}{x} \cdot \frac{x_0 - x}{x_0 x} = -\frac{1}{x_0 x}.
\]

Thus,

\[ f'(x_0) = \lim_{x \to x_0} \left( -\frac{1}{x_0 x} \right) = -\frac{1}{x_0^2}. \]
The difference quotient
\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{\frac{x_0 - x}{x_0 - x_0}}{x - x_0} = \frac{x_0 - x}{x_0} \cdot \frac{1}{x - x_0} = \frac{-x}{xx_0 (x - x_0)} = \frac{-1}{xx_0}.
\]

Thus,
\[
f'(x_0) = \lim_{x \to x_0} \frac{-1}{xx_0} = \frac{-1}{x_0}.
\]

Using formulation (2) of the definition of the derivative, we have the difference quotient
\[
\frac{f(x_0 + h) - f(x_0)}{h} = \frac{\frac{x_0 + h - x_0}{h}}{\frac{x_0 - (x_0 + h)}{(x_0 + h)x_0}} = \frac{-h}{(x_0 + h)x_0} \cdot \frac{1}{h} = \frac{-1}{(x_0 + h)x_0}.
\]

Thus,
\[
f'(x_0) = \lim_{h \to 0} \frac{-1}{(x_0 + h)x_0} = \frac{-1}{(x_0 + 0)x_0} = \frac{-1}{x_0^2}.
\]

5. We want to establish that the derivative of \( f(x) = mx + b \) (where \( m \) and \( b \) are assumed to be constants) is \( f'(x) = m \).

To do this, first let \( x_0 \) be a fixed but arbitrary real number.

Using formulation (1) of the definition of the derivative, we have the dif-
ference quotient
\[ \frac{f(x) - f(x_0)}{x - x_0} = \frac{(mx + b) - (mx_0 + b)}{x - x_0} \]
\[ = \frac{mx - mx_0}{x - x_0} \]
\[ = \frac{m(x - x_0)}{x - x_0} \]
\[ = m \]

Thus,
\[ f'(x_0) = \lim_{x \to x_0} m = m. \]

Using formulation (2) of the definition of the derivative, we have the difference quotient
\[ \frac{f(x_0 + h) - f(x_0)}{h} = \frac{(m(x_0 + h) + b) - (mx_0 + b)}{h} \]
\[ = \frac{mh}{h} \]
\[ = m \]

Thus,
\[ f'(x_0) = \lim_{h \to 0} m = m. \]

2 Exercise Set 4

1.

<table>
<thead>
<tr>
<th>( h )</th>
<th>approximate value of ( \frac{\sin h}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.99833</td>
</tr>
<tr>
<td>0.01</td>
<td>0.99998</td>
</tr>
<tr>
<td>0.00045</td>
<td>0.9999999662</td>
</tr>
<tr>
<td>-0.013</td>
<td>0.9999718338</td>
</tr>
<tr>
<td>-0.0046</td>
<td>0.999964733</td>
</tr>
<tr>
<td>-0.00004</td>
<td>pretty darn close to 1</td>
</tr>
</tbody>
</table>

3 Exercise Set 5

1.

<table>
<thead>
<tr>
<th>( h )</th>
<th>approximate value of ( \frac{\cos h - 1}{h} )</th>
</tr>
</thead>
<tbody>
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<td>-0.049958347</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.00499996</td>
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</tr>
<tr>
<td>-0.00004</td>
<td>pretty darn close to 0</td>
</tr>
</tbody>
</table>

3
1. For \( f(x) = \sin x \), we have \( f'(x) = \cos x \). Thus, \( f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = 1/2 \). Also, \( f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \sqrt{3}/2 \). Thus the tangent line to the graph of \( f \) at the point \( \left(\frac{\pi}{3}, \sqrt{3}/2\right) \) has equation

\[
y - \frac{\sqrt{3}}{2} = \frac{1}{2} \left( x - \frac{\pi}{3} \right)
\]

or

\[
y = \frac{1}{2} \left( x - \frac{\pi}{3} \right) + \frac{\sqrt{3}}{2}.
\]

The graph of \( f \) and this tangent line are shown below.

3. Since

\[
\frac{d}{dx} (\sin x) = \cos x
\]

and \(-\sin x = -1 \cdot \sin x\), the Constant Multiple Rule gives us

\[
\frac{d}{dx} (-\sin x) = \frac{d}{dx} (-1 \cdot \sin x)
= -1 \cdot \frac{d}{dx} (\sin x)
= -1 \cdot \cos x
= -\cos x.
\]

Since

\[
\frac{d}{dx} (\cos x) = -\sin x
\]
and \( -\cos x = -1 \cdot \cos x \), the Constant Multiple Rule gives us

\[
\frac{d}{dx} (-\cos x) = \frac{d}{dx} (-1 \cdot \cos x) \\
= -1 \cdot \frac{d}{dx} (\cos x) \\
= -1 \cdot (-\sin x) \\
= \sin x.
\]

5. (a) For \( y = 3 \sin x + x \), we have \( dy/dx = 3 \cos x + 1 \).

\[
\begin{align*}
\text{Graph of } y = 3 \sin x + x \\
\text{Graph of } dy/dx = 3 \cos x + 1
\end{align*}
\]

c. First, we write \( y = \cos (x + \pi/3) \) as

\[
y = \cos (\pi/3) \cdot \cos x - \sin (\pi/3) \cdot \sin x = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x.
\]

This gives us

\[
\frac{dy}{dx} = \frac{1}{2} (-\sin x) - \frac{\sqrt{3}}{2} \cdot \cos x = -\frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x.
\]
Graph of $y = \cos (x + \pi/3)$

Graph of $\frac{dy}{dx} = -\frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x$

(Note that $\frac{dy}{dx} = -\sin (x + \pi/3)$.)

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