Local Linear Approximation of Nonlinear Functions

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We now turn our attention to the problem of estimating the “slope” of a nonlinear function, $f$, at a point, $x_0$, in the domain of $f$. Another way to state this problem is to say that we are given a nonlinear function, $f$, and a point, $x_0$, in the domain of $f$, and we would like to find a linear function, $L$, such that $f(x) \approx L(x)$ for values of $x$ near $x_0$. The symbol “$\approx$” means “approximately equal to” and is, admittedly, somewhat ambiguous. Likewise, the phrase “values of $x$ near $x_0$” is also ambiguous. How close is “near”? These ambiguities will be eventually removed as we proceed in our work. Let us begin, though, by trying to get an intuitive understanding of the problem that we are addressing. Consider the graphs in Figures 1, 2, and 3:

![Graph](image-url)

Figure 1
At first glance, it appears that the graph in Figure 1 is the graph of a nonlinear function and that the graphs in Figures 2 and 3 are graphs of linear functions. However, the graphs in Figures 1, 2, and 3 are all graphs of the same nonlinear function! The difference is that the picture in Figure 1 shows the graph of this function, $f$, on the interval $[-3, 3]$ (and it is obvious from this picture that $f$ is a nonlinear function); whereas the pictures in Figures 2 and 3 show much smaller (or “zoomed-in”) graphs of $f$. In particular, Figure 2 shows the graph of $f$ on the interval $[-0.2, 0.2]$ and Figure 3 shows the graph of $f$ on the interval $[1.2, 1.4]$. Since the intervals over which $f$ is graphed in Figures 2 and 3 are quite small, it is almost impossible to see that $f$ is a nonlinear function by looking at these graphs. Figure 4 shows the same graph of $f$ as in Figure 1, but with two boxes indicating the parts of the graph of $f$ that are shown in Figures 2 and 3.

The point of this example is that if we “zoom in” on any one particular point, $(x_0, f(x_0))$, on the graph of $f$, we see a graph that is approximately
a line. Of course, the “approximate line” that we see depends on the point 
\((x_0, f(x_0))\) that we choose to zoom in on. For example, by looking at the 
graph in Figure 2, we observe that \(f\) is approximately equal to a linear 
function with slope 1 for values of \(x\) near \(x_0 = 0\). Likewise, by looking at 
the graph in Figure 3, see that \(f\) is approximately equal to a linear function 
with negative slope for values of \(x\) near \(x_0 = 1.3\).

1 Graphical and Numerical Estimates of Slopes 
of Linear Approximations

The fundamental idea that is needed to estimate the slope of the graph of 
a nonlinear function, \(f\), at a point \(x_0\) is quite simple. We choose a point \(x\) 
that is near (but not equal) to \(x_0\) and compute the slope of the line segment 
joining the points \((x_0, f(x_0))\) and \((x, f(x))\), both of which lie on the graph 
of \(f\). (See Figure 5.) This line segment is called the secant line joining 
\((x_0, f(x_0))\) and \((x, f(x))\).

\[ \text{graph of nonlinear} \]
\[ \text{function } f \]
\[ \text{dotted line is} \]
\[ \text{secant line joining points} \]
\[ (x_0, f(x_0)) \text{ and } (x, f(x)) \]

Figure 5

As an example, let us estimate the slope of the graph of the function
\( f(x) = x^2 - 4x + 2 \) at the point \( x_0 = 1 \). The graph of \( f \) is shown in Figure 6.

![Graph of \( f(x) = x^2 - 4x + 2 \)](image)

**Figure 6:** Graph of \( f(x) = x^2 - 4x + 2 \)

Since \( f(1) = -1 \), we will first examine the graph of \( f \) in a small square viewing window centered at the point \((1, -1)\). Such a viewing window can be obtained by moving a distance of 0.1 in each direction (left, right, up, and down) away from the point \((1, -1)\). On our graphing device, we set

\[
X_{\text{min}} = 1 - 0.1 = 0.9 \\
X_{\text{max}} = 1 + 0.1 = 1.1 \\
Y_{\text{min}} = -1 - 0.1 = -1.1 \\
Y_{\text{max}} = -1 + 0.1 = -0.9
\]

The graph of \( f \) in this viewing window is shown in Figure 7 and, for reference, Figure 8 shows a wider-view graph of \( f \) with a square box indicating the viewing window for the graph in Figure 7.
Although the graph shown in Figure 7 looks like the graph of a line, it is not. The graph is curved, but so slightly as to be imperceptible to the human eye. Since we know that the point \((x_0, f(x_0)) = (1, -1)\) is on this graph, we can estimate the slope of this “approximate line” by choosing any point, \((x, f(x))\), on the graph of \(f\) that is very close to the point \((x_0, f(x_0))\) and by then computing the slope of the secant line joining \((x_0, f(x_0))\) and \((x, f(x))\). Choosing \(x = 1.02\), we obtain

\[
f(x) = (1.02)^2 - 4(1.02) + 2 = -1.0396
\]

which means that the point \((x, f(x)) = (1.02, -1.0396)\) is on the graph of \(f\). Figure 9 shows the same graph as in Figure 7 and indicates the location of the points \((x_0, f(x_0)) = (1, -1)\) and \((x, f(x)) = (1.02, -1.0396)\).
The slope of the secant line joining the points \((x_0, f(x_0))\) and \((x, f(x))\) is

\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{-1.0396 - (-1)}{1.02 - 1} = \frac{-0.0396}{0.02} = -1.98.
\]

We have thus estimated that the slope of the graph of \(f(x) = x^2 - 4x + 2\) at the point \(x_0 = 1\) is approximately \(-1.98\).

Of course, the estimate of \(-1.98\) that we obtained was a result of our choice of the point \((x, f(x)) = (1.02, -1.0396)\) used in the calculation. If we choose some different point \((x, f(x))\) near \((x_0, f(x_0))\), we will obtain a different estimate because the graph of \(f\) is not a line. However, since the graph of \(f\) (shown in Figure 7) is approximately a line, the value of

\[
\frac{f(x) - f(x_0)}{x - x_0}
\]

should not vary too drastically as we vary the choice of \(x\), as long as we keep \(x\) close to \(x_0\). To test this claim, let us choose \(x = 0.99\) for which the corresponding value of \(f\) is \(f(0.99) = -0.9799\). The slope of the secant line joining the points \((x_0, f(x_0)) = (1, -1)\) and \((x, f(x)) = (0.99, -0.9799)\) is

\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{-0.9799 - (-1)}{0.99 - 1} = \frac{0.0201}{-0.01} = -2.01.
\]

Figure 10 shows the graph of \(f\) indicating the locations of the points used in this calculation.
As expected the estimate of $-2.01$ that we obtained in our second calculation is very close to the estimate of $-1.98$ obtained in our first calculation.

The table below shows the slopes of secant lines joining the points $(x_0, f(x_0)) = (1, -1)$ and $(x, f(x))$ for several choices of points $x$ close to $x_0 = 1$. (You should do the necessary calculations to make sure that you understand how these slopes were computed.)

<table>
<thead>
<tr>
<th>point $(x, f(x))$</th>
<th>slope of secant line joining $(1, -1)$ and $(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1.02, -1.0396)$</td>
<td>$-1.98$</td>
</tr>
<tr>
<td>$(0.99, -0.9799)$</td>
<td>$-2.01$</td>
</tr>
<tr>
<td>$(1.004, -1.007984)$</td>
<td>$-1.996$</td>
</tr>
<tr>
<td>$(0.9993, -0.99859951)$</td>
<td>$-2.0007$</td>
</tr>
<tr>
<td>$(1.0001, -1.00019999)$</td>
<td>$-1.9999$</td>
</tr>
<tr>
<td>$(0.9999, -0.99979999)$</td>
<td>$-2.0001$</td>
</tr>
</tbody>
</table>

Table 1

Based on the results in the table, it seems that if we were asked to assign a single number to be called the slope of $f$ at $x_0 = 1$, it would be reasonable to assign the value of $-2$ (since all of the slopes of the secant lines that were computed are close to $-2$). The actual line having slope $-2$ and passing through the point $(x_0, f(x_0)) = (1, -1)$ has equation

$$y - (-1) = -2(x - 1)$$

which can also be written as

$$y = -2x + 1.$$
We conclude that for values of $x$ near $x_0 = 1$, $f(x) \approx L(x)$ where $L$ is the linear function $L(x) = -2x + 1$. The line $y = -2x + 1$ is called the tangent line to the graph of $f$ at the point $(x_0, f(x_0)) = (1, -1)$. Figures 11 shows wide-view graphs of $f$ and $L$. Figure 12 shows zoomed-in graphs of both functions. Note that in Figure 12 the two graphs are barely distinguishable.

![Figure 11](image1.png)

Figure 11

![Figure 12](image2.png)

Figure 12

Guided by the preceding example, we outline a graphical/numerical procedure for estimating the slope of a nonlinear function $f$ at a point $x_0$.

**Procedure for Graphical/Numerical Estimation of Slope**

1. To get an overall visual picture of $f$, produce a “wide-view” graph of $f$ over a fairly large interval that contains the point $x_0$. 
2. By looking at the wide-view graph of \( f \), choose a small enough value of \( r > 0 \) such that the graph of \( f \) looks to be “almost linear” in the viewing window

\[
X_{\text{min}} = x_0 - r \\
X_{\text{max}} = x_0 + r \\
Y_{\text{min}} = f(x_0) - r \\
Y_{\text{max}} = f(x_0) + r
\]

Steps 1 and 2 are mainly for visualization.

3. Choose a value of \( x \) very close (but not equal) to \( x_0 \). A good guideline as to “how close” you should choose \( x \) to \( x_0 \) is that you should ensure that the point \((x, f(x))\) appears in the viewing window that you determined in step 2 (in which the graph of \( f \) appears to be “almost linear”).

4. Compute the slope of the secant line joining the points \((x_0, f(x_0))\) and \((x, f(x))\), i.e., compute

\[
\frac{f(x) - f(x_0)}{x - x_0}
\]

5. Repeat steps 3 and 4 using several different choices of \( x \) and make a table of slopes (like Table 1).

6. By examining the table that you made, choose a single number, \( m \), as your estimate of the slope of \( f \) at the point \( x_0 \). State your result using appropriate terminology such as: “I estimate that the slope of the graph of \( f \) at \( x_0 \) is \( m \).”

7. Write the equation of the tangent line to the graph of \( f \) at the point \((x_0, f(x_0))\). The tangent line has equation

\[
y - f(x_0) = m(x - x_0).
\]

Conclude that for values of \( x \) near \( x_0 \), \( f(x) \) is approximately equal to the linear function

\[
L(x) = m(x - x_0) + f(x_0).
\]

Finally, graph both \( f \) and \( L \) in your small viewing window \([x_0 - r, x_0 + r] \times [f(x_0) - r, f(x_0) + r]\). You should observe that the graphs are very
close to each other (perhaps so close that you can’t distinguish between the two). If the graphs are not close, then you probably did something wrong. Look over your work for mistakes!

**Example 1** In this example, we use the steps outlined in the above procedure to estimate the slope of the function $f(x) = \sqrt{x}$ at $x_0 = 1$.

1. “Wide-View” Graph of $f(x) = \sqrt{x}$

2. We observe that $f(x_0) = f(1) = 1$ (which means that the point (1,1) is on the graph of $f$) and that the graph of $f$ appears to be “almost linear” in the “zoomed-in” viewing window $[0.8, 1.2] \times [0.8, 1.2]$.

3-4. Choosing $x = 1.1$, the corresponding value of $f$ is $f(x) = f(1.1) = \sqrt{1.1}$. The slope of the secant line joining $(x_0, f(x_0))$ and $(x, f(x))$ is

$$\frac{\sqrt{1.1} - 1}{1.1 - 1} \approx 0.4880884817.$$
5. Choosing several different values of $x$ close to $x_0 = 1$, we compute slopes of secant lines as given in the following table:

<table>
<thead>
<tr>
<th>point $(x, f(x))$</th>
<th>slope of secant line joining $(1, 1)$ and $(x, f(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1.1, \sqrt{1.1})$</td>
<td>$0.4880884817$</td>
</tr>
<tr>
<td>$(0.9, \sqrt{0.9})$</td>
<td>$0.5131670195$</td>
</tr>
<tr>
<td>$(1.05, \sqrt{1.05})$</td>
<td>$0.4939015319$</td>
</tr>
<tr>
<td>$(0.992, \sqrt{0.992})$</td>
<td>$0.5010040201$</td>
</tr>
<tr>
<td>$(1.0006, \sqrt{1.0006})$</td>
<td>$0.4999250225$</td>
</tr>
<tr>
<td>$(0.9999, \sqrt{0.9999})$</td>
<td>$0.5000125006$</td>
</tr>
</tbody>
</table>

6. Based on the slope data given in the table, we estimate that the slope of the graph of $f(x) = \sqrt{x}$ at the point $x_0 = 1$ is 0.5.

7. The equation of the tangent line to the graph of $f$ at the point $(1,1)$ is

$$y - 1 = 0.5(x - 1)$$

which can be written as

$$y = 0.5x + 0.5.$$  

Thus, for values of $x$ near $x_0 = 1$, $f(x)$ is approximately equal to the linear function $L(x) = 0.5x + 0.5$. The figure below shows graphs of $f$ and $L$ in the viewing window $[0.8, 1.2] \times [0.8, 1.2]$. The graphs are very close to each other as expected.

![Graph showing linear approximation](image-url)
Exercise 2  1. For each function \( f \) and point \( x_0 \) given in parts a–l, follow the outlined procedure to estimate the slope of \( f \) at \( x_0 \). Follow the procedure in detail! In particular, include your graphs and use at least five or six different choices of \( x \) in your table of slopes of secant lines.

(a) \( f(x) = x^2 - 4x + 2, \quad x_0 = 3 \)
(b) \( f(x) = x^2 - 4x + 2, \quad x_0 = -1 \)
(c) \( f(x) = \sqrt{x}, \quad x_0 = 4 \)
(d) \( f(x) = \sqrt{x}, \quad x_0 = 0.16 \)
(e) \( f(x) = x^3 - 12x, \quad x_0 = 0 \)
(f) \( f(x) = x^3 - 12x, \quad x_0 = -4 \)
(g) \( f(x) = 1/x, \quad x_0 = 2 \)
(h) \( f(x) = 2^x, \quad x_0 = 0 \)
(i) \( f(x) = \sin x, \quad x_0 = \pi/3 \)
(j) \( f(x) = \cos x, \quad x_0 = 0 \)
(k) \( f(x) = x \cos(x^2), \quad x_0 = 0 \)
(l) \( f(x) = x \cos(x^2), \quad x_0 = 1.3 \)

2. Try to use the outlined procedure to estimate the slope of \( f(x) = 4 - |x| \) at the point \( x_0 = 0 \). What goes wrong?

3. Try to use the outlined procedure to estimate the slope of \( f(x) = x^{1/3} \) at the point \( x_0 = 0 \). What goes wrong?