Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator on this exam but you may not use any books or notes.

This exam contains 7 questions but you only need to do 5 of them. You can choose any 5. Even if you work on more than 5 of them, I will only grade 5 of them. You must choose which 5 you want me to grade. Below, circle the numbers of the 5 questions that you want me to grade.

1 2 3 4 5 6 7

1. For the function $f(x) = \sqrt{x}$ on the interval $[a, b] = [0, 9]$,

(a) Find the slope, $m$, of the secant line joining the points $(a, f(a))$ and $(b, f(b))$.

(b) Find a point $c$ in the interval $(a, b)$ such that $f'(c) = m$.

(c) Graph $f$ together with its tangent line at the point $(c, f(c))$. Also include, in this same picture, the secant line joining the points $(a, f(a))$ and $(b, f(b))$.

Solution:

a. $f(0) = \sqrt{0} = 0$ and $f(9) = \sqrt{9} = 3$. The slope of the secant line joining the points $(0, 0)$ and $(9, 3)$ is

$$m = \frac{3 - 0}{9 - 0} = \frac{1}{3}.$$

b. Note that $f(x) = x^{1/2}$ so

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Solving

$$\frac{1}{2\sqrt{x}} = \frac{1}{3}$$

we obtain

$$2\sqrt{x} = 3$$

which gives

$$\sqrt{x} = \frac{3}{2}$$
which gives

\[ x = \frac{9}{4}. \]

Note that \( c = 9/4 \) is in the interval \([0, 9]\). The equation of the tangent line to the graph of \( f(x) = \sqrt{x} \) at the point \( \left( \frac{9}{4}, f \left( \frac{9}{4} \right) \right) \) is

\[ y - \frac{3}{2} = \frac{1}{3} \left( x - \frac{9}{4} \right) \]

which can be simplified to

\[ y - \frac{3}{2} = \frac{1}{3} x - \frac{3}{4} \]

or

\[ y = \frac{1}{3} x + \frac{3}{4}. \]

c. The equation of the secant line joining the points \((0, 0)\) and \((9, 3)\) is

\[ y = \frac{1}{3} x. \]

The picture below shows that graph \( f(x) = \sqrt{x} \), it’s tangent line at the point \( \left( \frac{9}{4}, \frac{3}{2} \right) \), and the secant line joining the points \((0, 0)\) and \((9, 3)\).

2. In parts a, b, and c below, find the family of antiderivatives of the given function \( f \). State your results using complete sentences of the form “The family of antiderivatives of the function \( f(x) = \ldots \) consists of all functions of the form \( F(x) = \ldots \).”

(a) \( f(x) = x^6 \)
(b) \( f(x) = \sin(x) - \cos(x) \)
(c) \( f(x) = 3e^{6x} + 2x + 1 \)

Answers:
a. The family of antiderivatives of the function \( f(x) = x^6 \) consists of all functions of the form \( F(x) = \frac{1}{7}x^7 + C \) (where \( C \) can be any constant).

b. The family of antiderivatives of the function \( f(x) = \sin(x) - \cos(x) \) consists of all functions of the form \( F(x) = -\cos(x) - \sin(x) + C \) (where \( C \) can be any constant).

c. The family of antiderivatives of the function \( f(x) = 3e^{6x} + 2x + 1 \) consists of all functions of the form \( F(x) = \frac{1}{2}e^{6x} + x^2 + x + C \) (where \( C \) can be any constant).

3. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table below. Find lower and upper estimates for the distance that she travelled during these three seconds. (You must show your calculations in detail and include sentences that explain what you are doing and what your conclusions are.)

<table>
<thead>
<tr>
<th>( t ) (seconds)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (feet/sec)</td>
<td>0</td>
<td>6.2</td>
<td>10.8</td>
<td>14.9</td>
<td>18.1</td>
<td>19.4</td>
<td>20.2</td>
</tr>
</tbody>
</table>

**Solution:** Since the runner’s speed is always increasing, we can get a lower estimate of the total distance travelled by using the speed at the beginning of each 0.5 second interval (and assuming this to be the constant speed throughout the 0.5 second interval). Thus a lower estimate of the distance travelled is

\[
(0 \text{ ft/s})(0.5 \text{ s}) + (6.2 \text{ ft/s})(0.5 \text{ s}) + (10.8 \text{ ft/s})(0.5 \text{ s}) \\
+ (14.9 \text{ ft/s})(0.5 \text{ s}) + (18.1 \text{ ft/s})(0.5 \text{ s}) + (19.4 \text{ ft/s})(0.5 \text{ s}) \\
= 34.7 \text{ ft}
\]

and (by similar reasoning) an upper estimate of the distance travelled is

\[
(6.2 \text{ ft/s})(0.5 \text{ s}) + (10.8 \text{ ft/s})(0.5 \text{ s}) + (14.9 \text{ ft/s})(0.5 \text{ s}) \\
+ (18.1 \text{ ft/s})(0.5 \text{ s}) + (19.4 \text{ ft/s})(0.5 \text{ s}) + (20.2 \text{ ft/s})(0.5 \text{ s}) \\
= 44.8 \text{ ft}
\]

4. Use the definition of the definite integral

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} R_n
\]

to show that

\[
\int_2^3 x^2 \, dx = \frac{19}{3}.
\]

You must include all details of your work.

**Note:** Even though it is easy to use the Evaluation Theorem (Fundamental Theorem of Calculus, Part II) on this problem, that is not the point of the problem. You must use the definition of the integral.
Solution: We have (for any integer $n \geq 1$)

$$(\Delta x)_n = \frac{3 - 2}{n} = \frac{1}{n}$$

and for any integer $i$ (with $1 \leq i \leq n$)

$$x_i = 2 + i \cdot \frac{1}{n} = 2 + \frac{i}{n}$$

$$f(x_i) = x_i^2 = \left(2 + \frac{i}{n}\right)^2 = 4 + \frac{4i}{n} + \frac{i^2}{n^2}$$

$$f(x_i)(\Delta x)_n = \left(4 + \frac{4i}{n} + \frac{i^2}{n^2}\right) \cdot \frac{1}{n} = \frac{4}{n} + \frac{4i}{n^2} + \frac{i^2}{n^3}$$

and

$$R_n = \sum_{i=1}^{n} f(x_i)(\Delta x)_n$$

$$= \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{4i}{n^2} + \frac{i^2}{n^3}\right)$$

$$= \frac{4}{n} \sum_{i=1}^{n} 1 + \frac{4}{n^2} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{4}{n} \cdot n + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} + \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= 4 + 2 \left(\frac{n+1}{n}\right) + \frac{1}{6} \left(\frac{n+1}{n}\right) \left(\frac{2n+1}{n}\right)$$

$$= 4 + 2 \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).$$

Using the fact that $\lim_{n \to \infty} \frac{1}{n} = 0$, we obtain

$$\int_{2}^{3} x^2 \, dx = \lim_{n \to \infty} R_n$$

$$= \lim_{n \to \infty} \left(4 + 2 \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)\right)$$

$$= 4 + 2 (1 + 0) + \frac{1}{6} (1 + 0) (2 + 0)$$

$$= \frac{19}{3}.$$

5. The Evaluation Theorem (Fundamental Theorem of Calculus, Part II) tells us that if the function $f$ is continuous on the interval $[a, b]$ and if $F$ is an antiderivative of $f$ on $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a).$$
For example, since \( f(x) = x^2 \) is continuous on \([2, 3]\) and \( F(x) = \frac{1}{3}x^3 \) is an antiderivative of \( f \) on \([2, 3]\), then we conclude that

\[
\int_{2}^{3} x^2 \, dx = \frac{1}{3} (3^3) - \frac{1}{3} (2^3) = \frac{19}{3}.
\]

(a) Find an antiderivative of the function \( f(x) = 2x^3 - 4x^2 + 3x - 2 \) and then use the Evaluation Theorem to show that

\[
\int_{-1}^{2} (2x^3 - 4x^2 + 3x - 2) \, dx = -6.
\]

(b) The graph of \( f(x) = 2x^3 - 4x^2 + 3x - 2 \) on the interval \([-1, 2]\) is shown below. Explain why it “makes sense”, in terms of the graph that is shown, that the above definite integral is equal to a negative number. It would be helpful to shade certain parts of the given picture and to refer to these shadings in your explanation. (Write in complete sentences.)

Graph of \( f(x) = 2x^3 - 4x^2 + 3x - 2 \)

**Solution:** An antiderivative of \( f(x) = 2x^3 - 4x^2 + 3x - 2 \) is

\[
F(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 - 2x.
\]

Thus

\[
\int_{-1}^{2} (2x^3 - 4x^2 + 3x - 2) \, dx = F(2) - F(-1)
\]

\[
= \left( \frac{1}{2} (2)^4 - \frac{4}{3} (2)^3 + \frac{3}{2} (2)^2 - 2 (2) \right) - \left( \frac{1}{2} (-1)^4 - \frac{4}{3} (-1)^3 + \frac{3}{2} (-1)^2 - 2 (-1) \right)
\]

\[
= -6.
\]
Referring to the picture below: Since the given integral gives us the area of the black region minus the area of the red region and since the red region is bigger than the black region, it makes sense that the value of the integral is negative.

6. Let $F$ be the function

$$F(x) = \int_{2}^{x} t^2 \sin(t) \, dt.$$ 

The derivative of $F$ is (circle the correct choice):

(a) $F'(x) = x^2 \cos(x) + 2x \sin(x) - 4 \cos(2) - 4 \sin(2)$
(b) $F'(x) = x^2 \cos(x) + 2x \sin(x)$
(c) $F'(x) = x^2 \sin(x) - 4 \sin(2)$
(d) $F'(x) = x^2 \sin(x)$
(e) None of the above choices is correct.

You must explain how you arrive at your answer. The grading of this problem will be as follows

<table>
<thead>
<tr>
<th>Work</th>
<th>Points Awarded</th>
</tr>
</thead>
<tbody>
<tr>
<td>correct choice not circled</td>
<td>0</td>
</tr>
<tr>
<td>correct choice circled but explanation not correct</td>
<td>2</td>
</tr>
<tr>
<td>correct choice circled and explanation correct</td>
<td>10</td>
</tr>
</tbody>
</table>

Provide your explanation here: The Fundamental Theorem of Calculus (Part I) tells us that if $f$ is continuous on some interval $[a, b]$ and if $F$ is defined for all $x$ in $[a, b]$ by

$$F(x) = \int_{a}^{x} f(t) \, dt,$$

then $F'(x) = f(x)$. This is exactly the setup of this problem with $a = 2$ and $f(t) = t^2 \sin(t)$, so $F'(x) = f(x) = x^2 \sin(x)$.

7. The graph of a function, $f$, on the interval $[-4, 4]$ is shown below.
Let $g$ be the function defined, for all $x$ in the interval $[-4, 4]$, by

$$g(x) = \int_{-4}^{x} f(t) \, dt.$$  

(a) Complete the following table for $g$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-4$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>1</td>
<td>2.5</td>
<td>4</td>
<td>5.5</td>
<td>5.5</td>
<td>4.5</td>
<td>3.5</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) Determine all intervals on which $g$ is increasing and all intervals on which $g$ is decreasing. Explain your reasoning (referring to the graph of $f$ in your explanation and writing in complete sentences which do not contain the word “it”).

(c) Determine all intervals on which $g$ is concave up and all intervals on which $g$ is concave down. (*Hint:* There is some interval on which $g$ has no concavity; that is, $g$ is linear on some interval.) Explain your reasoning (referring to the graph of $f$ in your explanation and writing in complete sentences which do not contain the word “it”).

(d) Based on your findings in parts a, b, and c above, draw the graph of $g$.

(e) The Fundamental Theorem of Calculus (Part I) tells us that $g'(x) = f(x)$. Does the graph that you drew in part d above support this fact? Explain.

**Solution:** Since $g'(x) = f(x)$ and $f(x) > 0$ on the interval $(-4, 1/2)$, we conclude that $g$ is increasing on the interval $(-4, 1/2)$. Since $f(x) < 0$ on the interval $(1/2, 4)$, we conclude that $g$ is decreasing on the interval $(1/2, 4)$.
Since \( g''(x) = f'(x) \) and \( f'(x) > 0 \) on the intervals \((-4, -3)\) and \((-2, -1)\), we conclude that \( g \) is concave up on these intervals. Since \( f'(x) < 0 \) on the intervals \((-3, -2)\), \((-1, 0)\), \((0, 1)\) and \((3, 4)\), we conclude that \( g \) is concave down on these intervals. Since \( f'(x) = 0 \) on the interval \((1, 3)\), \( g \) has no concavity on this interval. \( g \) is linear (with slope \(-1\)) throughout the interval \((1, 3)\). The graph of \( g \) is shown below.