Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator on this exam but you may not use any books or notes.

This exam contains 6 questions but you only need to do 5 of them. You can choose any 5. Even if you work on more than 5 of them, I will only grade 5 of them. You must choose which 5 you want me to grade. Below, circle the numbers of the 5 questions that you want me to grade.

1. For the function \( f(x) = x^2 - 2x - 8 \) on the interval \([0, 6]\),

(a) Find the slope, \( m \), of the secant line joining the points \((a, f(a))\) and \((b, f(b))\).

(b) Find a point \( c \) in the interval \((a, b)\) such that \( f'(c) = m \).

(c) Graph \( f \) together with its tangent line at the point \((c, f(c))\). Also include, in this same picture, the secant line joining the points \((a, f(a))\) and \((b, f(b))\).

Solution:

a. \( f(0) = -8 \) and \( f(6) = 16 \). The slope of the secant line joining the points \((0, -8)\) and \((6, 16)\) is

\[
m = \frac{16 - (-8)}{6 - 0} = 4.
\]

b. Note that

\[
f'(x) = 2x - 2.
\]

Solving

\[2x - 2 = 4\]

we obtain

\[x = 3.
\]

Note that \( c = 3 \) is in the interval \([0, 6]\). The equation of the tangent line to the graph of \( f(x) = x^2 - 2x - 8 \) at the point \((3, f(3)) = (3, -5)\) is

\[y - (-5) = 4(x - 3)\]

which can be simplified to

\[y + 5 = 4x - 12\]

or

\[y = 4x - 17.\]
c. The equation of the secant line joining the points (0, −8) and (6, 16) is
\[ y - 16 = 4(x - 6) \]
which can be written as
\[ y - 16 = 4x - 24 \]
or
\[ y = 4x - 8 \]
The picture below shows that graph \( f(x) = x^2 - 2x - 8 \), it’s tangent line at the point (3, −5), and the secant line joining the points (0, −8) and (6, 16).

2. In parts a, b, and c below, find the family of antiderivatives of the given function \( f \). State your results using complete sentences of the form “The family of antiderivatives of the function \( f(x) = \_\_\_\_\_\_ \) consists of all functions of the form \( F(x) = \_\_\_\_\_\_ \)”. 

(a) \( f(x) = x^4 \)
(b) \( f(x) = \frac{1}{x^2} \)
(c) \( f(x) = \frac{1+\cos^2(x)}{\cos^2(x)} \)
(d) \( f(x) = e^{4x} \)
(e) \( f(x) = e^{4x-7} \)

**Answers:**

a. The family of antiderivatives of the function \( f(x) = x^4 \) consists of all functions of the form \( F(x) = \frac{1}{5}x^5 + C \) (where \( C \) can be any constant).

b. Note that \( \frac{1}{x^4} \) can be written as \( x^{-4} \). The family of antiderivatives of the function \( f(x) = x^{-4} \) consists of all functions of the form
\[ F(x) = \frac{1}{-3}x^{-3} + C = -\frac{1}{3x^3} + C \]
(where \( C \) can be any constant).
c. Note that
\[
\frac{1 + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \sec^2(x) + 1.
\]
The family of antiderivatives of the function \( f(x) = \sec^2(x) + 1 \) consists of all functions of the form \( F(x) = \tan(x) + x + C \) (where \( C \) can be any constant).

d. The family of antiderivatives of the function \( f(x) = e^{4x} \) consists of all functions of the form \( F(x) = \frac{1}{4}e^{4x} + C \) (where \( C \) can be any constant).

e. The family of antiderivatives of the function \( f(x) = e^{4x-7} \) consists of all functions of the form \( F(x) = \frac{1}{4}e^{4x-7} + C \) (where \( C \) can be any constant).

3. Use the **area interpretation** of the definite integral to show that
\[
\int_{2}^{3} x \, dx = 2.5.
\]
You must include all details of your work. In particular, your solution must include a graph of the function \( f(x) = x \) illustrating how the above integral is interpreted in terms of areas.

**Note:** Even though it is easy to use the Evaluation Theorem (Fundamental Theorem of Calculus, Part II) on this problem, that is not the point of the problem. You must use the area interpretation of the integral.

**Solution:**

![Graph of f(x) = x](image)

The above picture shows the graph of \( f(x) = x \). Using basic geometry we see that the area of the region that lies below the graph and above the \( x \) axis on the interval \([2, 3]\) is 2.5. Thus
\[
\int_{2}^{3} x \, dx = 2.5.
\]

4. Use the Evaluation Theorem (Part II of the Fundamental Theorem of Calculus) to show that
\[
\int_{2}^{3} x \, dx = 2.5.
\]
Solution: An antiderivative of \( f(x) = x \) is \( F(x) = \frac{1}{2}x^2 \). Thus
\[
\int_2^3 x \, dx = F(3) - F(2) = \frac{1}{2}(3)^2 - \frac{1}{2}(2)^2 = 2.5.
\]

5. Let \( F \) be the function
\[
F(x) = \int_2^x \arctan(t) \, dt.
\]
The derivative of \( F \) is (circle the correct choice):

(a) \( F'(x) = \frac{1}{1 + (\frac{1}{x})^2} \)

(b) \( F'(x) = \frac{1}{1 + (\frac{1}{x})^2} - \frac{1}{5} \)

(c) \( F'(x) = \arctan \left( \frac{1}{x} \right) - \arctan(2) \)

(d) \( F'(x) = -\frac{1}{x^2} \arctan \left( \frac{1}{x} \right) \)

(e) None of the above choices is correct.

You must explain how you arrive at your answer. The grading of this problem will be as follows

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<th>Points Awarded</th>
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<tr>
<td>correct choice circled but explanation not correct</td>
<td>4</td>
</tr>
<tr>
<td>correct choice circled and explanation correct</td>
<td>20</td>
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</tbody>
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Provide your explanation here: Using the fact that
\[
\frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(t) \, dt \right) = f(b(x))b'(x) - f(a(x))a'(x)
\]
and that here we have \( f(t) = \arctan(t) \), \( a(x) = 2 \), \( a'(x) = 0 \), \( b(x) = \frac{1}{x} \), and \( b'(x) = -\frac{1}{x^2} \), we see that
\[
F'(x) = \arctan \left( \frac{1}{x} \right) \cdot \left( -\frac{1}{x^2} \right) - \arctan(2) \cdot (0) = -\frac{1}{x^2} \arctan \left( \frac{1}{x} \right).
\]

6. The velocity function of a car travelling on a straight road (over the time interval from \( t = 0 \) to \( t = 6 \)) is
\[
v(t) = 25 - t^2
\]
Find the total displacement and the total distance travelled by the car during the time interval \([0, 6]\). Your solution must be detailed! Write in complete sentences that clearly explain your reasoning.

Solution: Note that the velocity of the car at time \( t = 5 \) is 0. Also, the velocity is positive during the time interval \([0, 5]\) and negative during the time interval \((5, 6]\).
This means that the car is moving to the right during the time interval $[0, 5)$ and to the left during the time interval $(5, 6)$. Since
\[
\int_0^5 (25 - t^2) \, dt = 25t - \frac{1}{3} t^3 \bigg|_{t=0}^{t=5} = 25(5) - \frac{1}{3} (5)^3 = \frac{250}{3}
\]
and
\[
\int_5^6 (25 - t^2) \, dt = 25t - \frac{1}{3} t^3 \bigg|_{t=5}^{t=6} = \left(25(6) - \frac{1}{3} (6)^3\right) - \frac{250}{3} = -\frac{16}{3},
\]
we see that the total displacement is
\[
\int_0^6 v(t) \, dt = \frac{250}{3} - \frac{16}{3} = 78
\]
and that the total distance travelled is
\[
\int_0^6 v(t) \, dt = \frac{250}{3} + \frac{16}{3} = 88.\overline{5}.
\]