1. \[ \lim_{{x \to 4}} f(x) = f(4). \]

3. Referring to the graph of \( f \) on page 128 of the textbook:
   a. \( f \) is not continuous at \(-2, 2, \) and \(4.
      \( f \) is not continuous at \(-2 \) because \( \lim_{{x \to -2}} f(x) \) does not exist.
      \( f \) is not continuous at \(2 \) because \( \lim_{{x \to 2}} f(x) \) does not exist.
      \( f \) is not continuous at \(4 \) because \( \lim_{{x \to 4}} f(x) \) does not exist.
      \textbf{Note}: \( f \) is not defined at \(-4, \) so the question of continuity (or lack thereof) does not apply at this point. However, the author of the textbook would interpret this to mean that \( f \) is not continuous at \(-4.\)
   b. \( f \) is continuous from the left, but not from the right, at \(-2.
      \( f \) is continuous from the right, but not from the left, at \(2.
      \( f \) is continuous from the right, but not from the left, at \(4.

5. The pictured function, \( f, \) is continuous everywhere except at \(3, \) but it is continuous from the left at \(3.

7. a.
b. This function has discontinuities at 1, 2, 3, and 4. To one who parks at this lot, it means that there is a big jump in price for parking, say, for just over 2 hours as compared to parking for just under 2 hours.

9. Suppose that \( f \) and \( g \) are continuous functions with \( f(3) = 5 \) and \( \lim_{x \to 3} (2f(x) - g(x)) = 4 \). Since \( f \) is continuous at 3, we know that
\[
\lim_{x \to 3} f(x) = f(3).
\]
Thus
\[
\lim_{x \to 3} f(x) = 5.
\]
Using the limit laws, we obtain
\[
\lim_{x \to 3} (2f(x) - g(x)) = 2 \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x)
\]
\[
= 2 \cdot 5 - \lim_{x \to 3} g(x).
\]
Thus
\[
10 - \lim_{x \to 3} g(x) = 4
\]
which means that
\[
\lim_{x \to 3} g(x) = 6.
\]
Since \( g \) is continuous at 3, we know that
\[
\lim_{x \to 3} g(x) = g(3).
\]
Therefore, \( g(3) = 6 \).

11. The function \( f(x) = (x + 2x^3)^4 \) is a polynomial function. It is continuous at all real numbers. To see that \( f \) is in fact a polynomial function, note that
\[
(x + 2x^3)^4 = x^4 + 8x^6 + 24x^8 + 32x^{10} + 16x^{12}.
\]

13. The function \( f(x) = \ln|x - 2| \) is not defined at \( x = 2 \) (because \( \ln(0) \) is not defined). Because the number 2 is not in the domain of \( f \), the continuity question is not relevant.
Here is the graph of \( f \). (It has a vertical asymptote at \( x = 2 \).)

15. The function being studied here is

\[
  f(x) = \begin{cases} 
    \frac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\
    -5 & \text{if } x = -3
  \end{cases}
\]

We note that

\[
  \frac{x^2 - x - 12}{x + 3} = \frac{(x + 3)(x - 4)}{x + 3} = x - 4 \text{ for all } x \neq -3.
\]

Thus, it is true that

\[
  f(x) = \begin{cases} 
    x - 4 & \text{if } x \neq -3 \\
    -5 & \text{if } x = -3
  \end{cases}
\]

From this, we see that

\[
  \lim_{x \to -3} f(x) = \lim_{x \to -3} (x - 4) = -7
\]

and

\[
  f(-3) = -5.
\]

Since \( \lim_{x \to -3} f(x) \neq f(-3) \), we conclude that \( f \) is not continuous at \(-3\).

Here is the graph of \( f \).

17. The domain of a function defined by the formula
\[
f(x) = \frac{x}{x^2 + 5x + 6}
\]
can contain any real number except \(-2\) and \(-3\). This is because
\[
\frac{x}{x^2 + 5x + 6} = \frac{x}{(x + 2)(x + 3)}
\]
meaning that \(x = -2\) or \(x = -3\) would make the denominator be 0.

Since \(f\) is a rational function (a ratio of two polynomials), it is continuous at all points in its domain.

19. The function \(f(x) = e^x\) is continuous at all real numbers because it is an exponential function. The function \(g(x) = \sin(x)\) is continuous at all points because it is a trigonometric function (with domain all real numbers). Also, the function \(h(x) = 5x\) is continuous at all real numbers because it is a polynomial (actually linear) function.

Now, if we let \(k(x) = \sin(5x)\), then \(k(x) = g(h(x))\). Since \(k\) is the composition of two functions that are both continuous at all real numbers, then \(k\) itself is continuous at all real numbers.

Finally, since \(e^x \sin(5x) = f(x) \cdot k(x)\), then this function is continuous at all real numbers because it is the product of two functions that are continuous at all real numbers.

21. The function \(g(t) = t^4 - 1\) is a polynomial function, and is hence continuous at all real numbers. The function \(f(t) = \ln(t)\) is a logarithmic function and is hence continuous at all real numbers \(t > 0\). (Recall that this function is only defined for \(t > 0\).)

Since \(G(t) = \ln(t^4 - 1) = f(g(t))\) (a composition), \(g\) is continuous at all points \(t > 0\) and \(f\) is continuous at all points, then \(G\) is continuous at all points \(t > 0\) (by Theorem 9 on page 126 of the textbook).

23. Consider the function defined by
\[
f(x) = \frac{1}{1 + e^{1/x}}.
\]
Clearly, the domain of this function cannot contain \(x = 0\) because there is a \(1/x\) in the formula and \(1/0\) is not defined. However, the domain can contain any number other than \(x = 0\). (There is no danger of dividing by \(0\) in the overall formula because \(e^{1/x} > 0\) for all \(x \neq 0\)).

Since \(g(x) = e^x\) is an exponential function, it is continuous everywhere. Also, \(h(x) = 1/x\) is continuous at all points in its domain (which is everything except \(x = 0\)). Thus the composition \(e^{1/x} = g(h(x))\) is continuous at all real numbers (except \(x = 0\) where the question is not relevant).

Since \(k(x) = 1\) is a polynomial (actually constant) function, it is continuous everywhere, and since \(1 + e^{1/x}\) is a sum of functions that are continuous throughout their domains, then it is continuous throughout its domain.

Finally, the original function, \(f\), is the ratio of two functions that are continuous throughout their domains, so \(f\) is continuous throughout its domain. In conclusion, \(f\) has domain consisting of all real numbers except 0, and \(f\) is continuous at each point in its domain.

25.
\[
\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}} = \frac{5 + \sqrt{4}}{\sqrt{5 + 4}} = \frac{5 + 2}{\sqrt{9}} = \frac{7}{3}.
\]
27.
\[
\lim_{x \to 1} e^{x^2-x} = e^{1^2-1} = e^0 = 1.
\]

29. For the function
\[
f(x) = \begin{cases} 
  x + 2 & \text{if } x < 0 \\
  e^x & \text{if } 0 \leq x \leq 1 \\
  2 - x & \text{if } x > 1 
\end{cases},
\]
the only points at which there arises a question about continuity is the points 0 and 1. This is because \( f \) is equal to a polynomial function on the intervals \((-\infty, 0)\) and \((1, \infty)\) and is equal to an exponential function on the interval \((0, 1)\).

Taking left and right hand limits of \( f \) at 0, we have
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 2) = 2
\]
and
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x = e^0 = 1.
\]
Since \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \) are not equal, we conclude that \( \lim_{x \to 0} f(x) \) does not exist and hence that \( f \) is not continuous at 0.

Taking left and right hand limits of \( f \) at 1, we have
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} e^1 = e \approx 2.71828
\]
and
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x) = 1.
\]
Since \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \) are not equal, we conclude that \( \lim_{x \to 1} f(x) \) does not exist and hence that \( f \) is not continuous at 1.

Here is the graph of \( f \):

![Graph of f(x) showing discontinuities at x=0 and x=1](image-url)
31. No matter what \( c \) is, the function
\[
f(x) = \begin{cases} 
  cx + 1 & \text{if } x \leq 3 \\
  cx^2 - 1 & \text{if } x > 3 
\end{cases}
\]
is continuous on the interval \((-\infty, 3)\) and on the interval \((3, \infty)\) because \( f \) is equal to a polynomial function on each of these intervals. The only point in question is \( x = 3 \). In order to have \( f \) be continuous at 3, we must have
\[
\lim_{x \to 3} f(x) = f(3).
\]
Since
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (cx + 1) = 3c + 1
\]
and
\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (cx^2 - 1) = 9c - 1,
\]
we must have
\[
3c + 1 = 9c - 1
\]
which means that \( c \) must equal 1/3. If we set \( c = 1/3 \), then
\[
f(3) = \frac{1}{3}(3) + 1 = 2
\]
and
\[
\lim_{x \to 3} f(x) = 2
\]
so \( f \) is continuous at 3.

33. Let \( f(x) = x^3 - x^2 + x \). Since \( f \) is a polynomial function, it is continuous everywhere.
   Now note that \( f(0) = 0 \) and \( f(3) = 21 \).
   Since the number 10 lies between the numbers 0 and 21, then the Intermediate Value Theorem tells us that there must be some number \( c \) lying between 0 and 3 such that \( f(c) = 10 \).

35. Let \( f(x) = x^3 - 3x + 1 \). Since \( f \) is a polynomial function, it is continuous everywhere.
   Now note that \( f(0) = 1 \) and \( f(1) = -1 \).
   Since the number 0 lies between the numbers 1 and -1, then the Intermediate Value Theorem tells us that there must be some number \( c \) lying between 0 and 1 such that \( f(c) = 0 \). This number \( c \) is a solution of the equation \( x^3 - 3x + 1 = 0 \).

37. Let \( f(x) = \cos(x) - x \). This function is continuous everywhere (because it is the difference of two functions that are continuous everywhere).
   Now note that \( f(0) = 1 \) and \( f(1) = \cos(1) - 1 \approx -0.45970 \).
   Since the number 0 lies between the numbers 1 and -0.45970, then the Intermediate Value Theorem tells us that there must be some number \( c \) lying between 0 and 1 such that \( f(c) = 0 \). This number \( c \) is a solution of the equation \( \cos(x) - x = 0 \).

45. Is there a number that is exactly one more that its cube? In other words, is there a number \( x \) such that \( x = x^3 + 1 \)? This is true if and only if there is a number \( x \) such that \( x - x^3 = 1 \).
   Let \( f(x) = x - x^3 \). This function is continuous everywhere because it is a polynomial function.
Now note that $f(0) = 0$ and $f(-2) = 6$.

Since the number 1 lies between the numbers 0 and 6, then the Intermediate Value Theorem tells us that there must be some number $c$ lying between 0 and $-2$ such that $f(c) = 1$. This number $c$ is a solution of the equation $x - x^3 = 1$.

We conclude that there is some number that is exactly one more than its cube (and that such a number can be found somewhere between 0 and $-2$).

47. This is a fun problem. Perhaps we will discuss it in class if somebody asks about it!