These notes closely follow the presentation of the material given in James Stewart’s textbook Calculus, Concepts and Contexts (2nd edition). These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

**Continuity**

**Definition** Let $f$ be a function whose domain contains the point $a$ and also contains points arbitrarily close to $a$. If

$$\lim_{x \to a} f(x) = f(a),$$

then $f$ is said to be **continuous** at $a$.

**Example** The graph of a function, $f$, is shown below. At which numbers is $f$ continuous? At which numbers is $f$ not continuous?
Example  At which points are each of the following functions continuous? At which points are they discontinuous? (Where not specified, assume that the domain of the function is the largest possible domain on which the given formula defines a function.)

1) \( f(x) = \frac{x^2 - x - 2}{x - 2} \)

2) \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

3) \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \)

4) \( f(x) = \lfloor x \rfloor \)
Definition Let \( f \) be a function whose domain contains the point \( a \) and also contains points to the left of \( a \) that are arbitrarily close to \( a \). If
\[
\lim_{x \to a^-} f(x) = f(a),
\]
then \( f \) is said to be **continuous from the left** at \( a \).

Definition Let \( f \) be a function whose domain contains the point \( a \) and also contains points to the right of \( a \) that are arbitrarily close to \( a \). If
\[
\lim_{x \to a^+} f(x) = f(a),
\]
then \( f \) is said to be **continuous from the right** at \( a \).

Example We have seen that the function \( f(x) = |x| \) is continuous at the point \( a \) if and only if \( a \) is not an integer. Suppose that \( a \) is an integer and decide whether or not \( f \) is left continuous (right continuous) at \( a \).
**Definition**  Suppose that $f$ is a function whose domain contains the interval $I$. If $f$ is continuous at each point in $I$, then $f$ is said to be **continuous on** $I$.

**Example**  Explain why the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

*Graph of $f(x) = 1 - \sqrt{1 - x^2}$*
**Theorem** If $f$ and $g$ are functions that are continuous at $a$ and if $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f - g$
3. $cf$
4. $fg$
5. $\frac{f}{g}$ (assuming that $g(a) \neq 0$)

**Theorem** The following classes of functions are all continuous at all points in their domains:

1. **polynomial functions** of the form $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
2. **rational functions** of the form $r(x) = p(x)/q(x)$ where $p$ and $q$ are polynomials
3. **root functions** of the form $f(x) = \sqrt[n]{x}$
4. **exponential functions** of the form $f(x) = a^x$ (where $a$ is a positive constant)
5. **logarithmic functions** of the form $f(x) = \log_a(x)$ (where $a$ is a positive constant)
6. **trigonometric functions** ($\sin, \cos, \tan, \cot, \sec, \csc$)
7. **inverse trigonometric functions** ($\arcsin, \arccos, \arctan, \arccot, \arcsec, \arccsc$)

**Example** Find

$$\lim_{x \to 2} \frac{x^3 - 9x}{x - 1}.$$
Example Find

\[ \lim_{x \to \frac{\pi}{3}} \tan(x). \]

Example Find

\[ \lim_{x \to \frac{\pi}{3}} \frac{3x - \sin(x)}{2 + \cos(x)}. \]
**Theorem**  If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right).$$

**Example**  Evaluate

$$\lim_{x \to 2} \arccos \left( \frac{1}{x} \right).$$
Theorem  If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at $a$.

(Recall that $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$.)

Example  At which points are each of the following functions continuous?

1. $F(x) = \sin(x^2)$
2. $F(x) = \ln(1 + \cos(x))$
The Intermediate Value Theorem

If a function, \( f \), is continuous on an interval \( I \) (meaning that \( f \) is continuous at each point of \( I \)), then the graph of \( f \) has no “holes” or “breaks” anywhere in the interval \( I \). This means that the graph of \( f \) must “pass through” every \( y \) value between any two \( y \) values that the graph is known to pass through. We state this fact more formally as the Intermediate Value Theorem.

**Theorem (Intermediate Value Theorem)** Suppose that \( a \) and \( b \) are real numbers with \( a < b \) and suppose that \( f \) is a function that is continuous on the interval \( [a,b] \). Also suppose that \( N \) is a real number that lies between \( f(a) \) and \( f(b) \). Then there exists a number \( c \), lying between \( a \) and \( b \), such that \( f(c) = N \).

**Example** Use the Intermediate Value Theorem to show that there is a solution of the equation

\[
4x^3 - 6x^2 + 3x - 2 = 0
\]

that lies between 1 and 2.

**Note:** We are not asked to actually find a solution of this equation. We are only asked to show that there is a solution.