Implicit Differentiation

A formula, $y = f(x)$, defines $y$ explicitly as a function of $x$. We say “explicitly” because $y$ is “solved for” in terms of $x$. Given any $x$ in the domain of $f$, we can use the formula $y = f(x)$ to compute the corresponding value of $y$.

More generally, a formula of the form $F(x, y) = 0$ defines a relation between $x$ and $y$. Sometimes, $F(x, y) = 0$ can be solved for $y$ in terms of $x$ and we arrive at an explicitly-defined function $y = f(x)$. Other times, however, it is not possible to solve explicitly for $y$ in terms of $x$. Even in these cases though, it might still be true that $y$ is a function of $x$ – meaning that each $x$ value corresponds to exactly one $y$ value that makes the equation $F(x, y) = 0$ be true. It might also happen that each $x$ value corresponds to more than one $y$ value that makes $F(x, y) = 0$ be true – in which case we do not have a function.

Even if the equation $F(x, y) = 0$ does not uniquely define $y$ as a function of $x$, it is still generally true that the graph of $F(x, y) = 0$ is a curve in the $x,y$–plane and we can still inquire about the slope of the tangent line to the graph of $F(x, y) = 0$ at a given point on the curve. The process of finding $\frac{dy}{dx}$ where $y$ is defined implicitly by an equation of the form $F(x, y) = 0$ is called implicit differentiation.
Example  Consider the equation $y^2 = x$. This equation defines a curve in the $x,y$–plane that is pictured below.

Graph of the curve $y^2 = x$

To find the slope of the tangent line to this curve at any point on the curve (except at the point $(0, 0)$ where the tangent line is vertical), we compute the derivative with respect to $x$ of both sides of the equation $y^2 = x$ treating $y$ as a function of $x$. Since we are treating $y$ as a function of $x$, the Chain Rule must be used.

\[
\frac{d}{dx}(y^2) = \frac{d}{dx}(x)
\]

\[
\Rightarrow 2y \cdot \frac{dy}{dx} = 1
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{1}{2y}.
\]

This tells us that the slope of the tangent line to the curve $y^2 = x$ at any point on the curve is simply $1/(2y)$. Note that this expression is not defined when $y = 0$. This value of $y$ corresponds to the point $(0, 0)$ on the curve, at which the tangent line is vertical.
Example  In the previous Example, we could have first solved the equation \( y^2 = x \) for \( y \) and then used explicit differentiation: Solving \( y^2 = x \) for \( y \) gives us

\[
y = \pm \sqrt{x}.
\]

Thus, we see that the curve \( y^2 = x \) is a combination of the graphs of two functions: \( y = \sqrt{x} \) and \( y = -\sqrt{x} \). Which explicit formula to use in finding the slope of a tangent line using explicit differentiation would depend on which point on the curve \( y^2 = x \) we want to find the slope of the tangent line. For example, if we wanted to find the slope of the tangent line to the curve \( y^2 = x \) at the point \((4, 2)\), then we would differentiate the formula \( y = \sqrt{x} \). On the other hand, if we wanted to find the slope of the tangent line to the curve \( y^2 = x \) at the point \((4, -2)\), then we would differentiate the formula \( y = -\sqrt{x} \).

As can be seen from this example, implicit differentiation is often more efficient because it gives us an answer (\( 1/(2y) \) in this case) that works no matter which point on the curve we are considering.
Example  Find an equation of the tangent line to the circle
\[ x^2 + y^2 = 25 \]
at the point (3,4). Do this in two ways: by using implicit differentiation and by using explicit differentiation.
Example The Folium of Descartes is the curve defined by the formula
\[ x^3 + y^3 = 6xy. \]
This curve is pictured below.

1. Use implicit differentiation to compute \( \frac{dy}{dx} \).
2. Find the slope of the tangent line to the Folium of Descartes at the point \((3, 3)\).
**Example** Compute $dy/dx$ for the curve

$$\sin(x + y) = y^2 \cos(x).$$