Derivatives of Logarithmic Functions

Recall that if $a$ is a positive number (a constant) with $a \neq 1$, then
\[ y = \log_a(x) \]
means that \[ a^y = x. \]

The function \( y = \log_a(x) \), which is defined for all \( x > 0 \), is called the base \( a \) logarithm function.

To find the derivative of the base $e$ logarithm function, \( y = \log_e(x) = \ln(x) \), we write the formula in the implicit form \( e^y = x \) and then take the derivative of both sides of this formula (with respect to $x$) in order to find $dy/dx$. Thus, we use implicit differentiation. The reason for first writing the formula as $e^y = x$ is because we already know how to differentiate the base $e$ exponential function. In fact, recall (from Section 3.1) that the derivative of the function
\[ u = e^y \]
is
\[ \frac{du}{dy} = e^y. \]

Writing
\[ e^y = x, \]
we differentiate both sides with respect to $x$ to obtain
\[ \frac{d}{dx}(e^y) = \frac{d}{dx}(x) \]
or
\[ \frac{d}{dx}(u) = \frac{d}{dx}(x) \]
where \( u = e^y \).

By the Chain Rule (recall that we are treating $y$ as a function of $x$), we obtain
\[ \frac{du}{dy} \cdot \frac{dy}{dx} = 1 \]
or
\[ e^y \cdot \frac{dy}{dx} = 1 \]
This gives us
\[ \frac{dy}{dx} = \frac{1}{e^y} \]
or (in terms of $x$ only),
\[ \frac{dy}{dx} = \frac{1}{x}. \]

In conclusion, the derivative function of the base $e$ logarithm function, \( y = \ln(x) \), is simply
\[ \frac{dy}{dx} = \frac{1}{x}. \]

In other words,
\[ \frac{d}{dx} (\ln(x)) = \frac{1}{x}. \]

If we are using a base other than base \(e\), say base \(a\), then we recall the “change of base formula” which tells us that
\[ \log_a(x) = \frac{\ln(x)}{\ln(a)}. \]

Since \(\ln(a)\) is a constant, we can use the Constant Multiple Rule to compute the derivative of \(y = \log_a(x)\). In particular,
\[
\frac{d}{dx} (\log_a(x)) = \frac{d}{dx} \left( \frac{\ln(x)}{\ln(a)} \right) \\
= \frac{1}{\ln(a)} \cdot \frac{d}{dx} (\ln(x)) \\
= \frac{1}{\ln(a)} \cdot \frac{1}{x}.
\]

In summary
\[ \frac{d}{dx} (\log_a(x)) = \frac{1}{\ln(a) \cdot x}. \]

Finally, let us return to the problem of finding the derivative of the base \(a\) exponential function, \(y = a^x\). Writing this as \(\log_a(y) = x\) and using implicit differentiation, we see that
\[
\frac{d}{dx} (\log_a(y)) = \frac{d}{dx} (x)
\]

which implies that
\[
\frac{1}{\ln(a) \cdot y} \cdot \frac{dy}{dx} = 1
\]

which implies that
\[ \frac{dy}{dx} = \ln(a) \cdot y \]

or
\[ \frac{dy}{dx} = \ln(a) \cdot a^x. \]

In other words,
\[ \frac{d}{dx} (a^x) = \ln(a) \cdot a^x. \]
To summarize all of this: If we have any base $a$ with $a > 0$ and $a \neq 1$, then

\[ \frac{d}{dx}(a^x) = \ln(a) \cdot a^x \]

\[ \frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a) \cdot x} \]

In the case of base $e$, the formulas for the derivatives are especially nice:

\[ \frac{d}{dx}(e^x) = e^x \]

\[ \frac{d}{dx}(\ln(x)) = \frac{1}{x} \]

Example  State the formulas for the derivatives of the base 2 exponential function, the base 3 exponential function, the base 2 logarithm function, and the base 3 logarithm function.
Logarithmic Differentiation

To find the derivatives of functions of the form

$$ y = (f(x))^g(x), $$

it is often easiest to first take the logarithm of both sides of the formula and to then compute the derivative using implicit differentiation. This process is called logarithmic differentiation.

**Example**  Find the derivative, $dy/dx$, of the function

$$ y = x^{\cos(x)} $$

whose (partial) graph is shown below.

![Graph of $y = x^{\cos(x)}$](image)

**Solution**  First we take the natural logarithm of both sides of the formula

$$ y = x^{\cos(x)} $$

to obtain

$$ \ln(y) = \ln(x^{\cos(x)}). $$

Using a property of logarithms, we obtain

$$ \ln(y) = \cos(x) \cdot \ln(x). $$

We now take the derivative with respect to $x$:

$$ \frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\cos(x) \cdot \ln(x)) $$

and use implicit differentiation to obtain

$$ \frac{1}{y} \cdot \frac{dy}{dx} = \cos(x) \cdot \frac{1}{x} + (-\sin(x)) \cdot \ln(x) $$

or
Writing the formula in terms of $x$ only, we see that
\[
\frac{dy}{dx} = x \cdot \left( \frac{\cos(x)}{x} - \sin(x) \cdot \ln(x) \right).
\]

Here is the graph of this derivative: