7. For the function \( f(x) = x^3 - 12x + 1 \), we have
\[
\begin{align*}
  f'(x) &= 3x^2 - 12 \\
  f''(x) &= 6x.
\end{align*}
\]

First we find the critical numbers by solving
\[3x^2 - 12 = 0.\]
The solutions of this equation are \( x = 2 \) and \( x = -2 \), so these are the critical numbers of \( f \). By looking at
\[f'(x) = 3(x + 2)(x - 2),\]
we see that the sign of \( f'(x) \) is as illustrated in the following diagram:

\[
\begin{array}{c|c|c|c}
  f''(x) > 0 & f'(x) < 0 & f'(x) > 0 \\
  -2 & 2 & x
\end{array}
\]

Thus
- \( f \) is increasing on the interval \((-\infty, -2)\).
- \( f \) has a local maximum value of \( f(-2) = 17 \) that occurs at \( x = -2 \).
- \( f \) is decreasing on the interval \((-2, 2)\).
- \( f \) has a local minimum value of \( f(2) = -15 \) that occurs at \( x = 2 \).
- \( f \) is increasing on the interval \((2, \infty)\).

To study concavity, we study the sign of \( f''(x) = 6x \). First, we note that the only solution of the equation
\[6x = 0\]
is \( x = 0 \). This means that there might be an inflection point of \( f \) at \( x = 0 \). The sign of \( f''(x) \) is as illustrated in the following diagram.
We now see that

- $f$ is concave down on $(-\infty, 0)$.
- $f$ has an inflection point at $x = 0$.
- $f$ is concave up on $(0, \infty)$.

A graph of $f$ is shown below. $x^3 - 12x + 1$

Graph of $f(x) = x^3 - 12x + 1$

15. We want to find the local maximum and minimum values of the function

$$f(x) = x + \sqrt{1-x} = x + (1-x)^{\frac{1}{2}}$$

and we want to do this in two ways: using the First Derivative Test and using the Second Derivative Test.

First, we observe that the function $f$ is defined only for all $x \leq 1$.

**Solution 1:** The first derivative of $f$ is

$$f'(x) = 1 + \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$= 1 - \frac{1}{2\sqrt{1-x}}$$

$$= \frac{2\sqrt{1-x}}{2\sqrt{1-x}} - \frac{1}{2\sqrt{1-x}}$$

$$= \frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}}.$$
\[
\frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}} = 0
\]

occur when
\[
2\sqrt{1-x} - 1 = 0.
\]

Solving this equation gives
\[
2\sqrt{1-x} = 1 \\
\Rightarrow \left(2\sqrt{1-x}\right)^2 = 1^2 \\
\Rightarrow 4(1-x) = 1 \\
\Rightarrow 1-x = \frac{1}{4} \\
\Rightarrow x = \frac{3}{4}.
\]

Thus, the only critical number of \(f\) is \(x = \frac{3}{4}\).

Now we note that
\[
f'\left(\frac{1}{2}\right) = \frac{2\sqrt{1-\frac{1}{2}} - 1}{2\sqrt{1-\frac{1}{2}}} \approx 0.29 > 0
\]

and
\[
f'\left(\frac{9}{10}\right) = \frac{2\sqrt{1-\frac{9}{10}} - 1}{2\sqrt{1-\frac{9}{10}}} \approx -0.58 < 0.
\]

These sample calculations show us that \(f\) is increasing on the interval \((-\infty, \frac{3}{4})\) and that \(f\) is decreasing on the interval \(\left(\frac{3}{4}, 1\right)\). Thus, \(f\) has a local maximum value of \(f\left(\frac{3}{4}\right) = \frac{5}{4}\) that occurs at \(x = \frac{3}{4}\).

A graph of \(f\) is shown below. \(f(x)\)

![Graph of f(x) = x + \sqrt{1-x}](image)

**Solution 2.** We have already done the work to show that \(f'\left(\frac{3}{4}\right) = 0\). The first derivative of \(f\) is

\[
f(x) = 1 - \frac{1}{2}(1-x)^{\frac{1}{2}}
\]
\[ f''(x) = \frac{1}{2} \cdot \frac{1}{2} (1 - x)^{-\frac{3}{2}} (-1) = \frac{-1}{4\sqrt{(1 - x)^3}}. \]

Since
\[ f'' \left( \frac{3}{4} \right) = \frac{1}{4\sqrt{\left(1 - \frac{3}{4}\right)^3}} = -2 < 0, \]
we conclude that \( f \) has a local maximum occurring at \( x = \frac{3}{4} \).

23. For the function \( f(x) = 2\cos(x) + \sin^2(x) \) with domain \([-\pi, \pi]\), we have
\[
\begin{align*}
  f'(x) &= -2\sin(x) + 2\sin(x)\cos(x) \\
       &= 2\sin(x)(\cos(x) - 1)
\end{align*}
\]
and
\[
\begin{align*}
  f''(x) &= 2\sin(x) \cdot (-\sin(x)) + 2\cos(x) \cdot (\cos(x) - 1) \\
         &= -2\sin^2(x) + 2\cos^2(x) - 2\cos(x) \\
         &= 2(\cos^2(x) - \sin^2(x) - \cos(x)) \\
         &= 2(2\cos^2(x) - \cos(x) - 1) \\
         &= 2(2\cos(x) + 1)(\cos(x) - 1).
\end{align*}
\]
To find the critical numbers of \( f \), we solve
\[ 2\sin(x)(\cos(x) - 1) = 0. \]
The solutions of this equation occur when either
\[ \sin(x) = 0 \]
or
\[ \cos(x) - 1 = 0. \]
The solutions of \( \sin(x) = 0 \) that lie in the interval \([-\pi, \pi]\) are \( x = -\pi, x = 0 \), and \( x = \pi \). However, we don’t consider the endpoints of the interval in question as possible critical numbers. Therefore, we have only found one critical number: \( x = 0 \).
The only solution of \( \cos(x) - 1 = 0 \), which is equivalent to \( \cos(x) = 1 \), that lies in the interval \([-\pi, \pi]\) is \( x = 0 \).
Therefore \( f \) has exactly one critical number in the interval \([-\pi, \pi]\). It is \( x = 0 \).
The value of \( f \) at \( x = 0 \)
\[ f(0) = 2\cos(0) + \sin^2(0) = 2(1) + (0)^2 = 2. \]
Now note that if \( x \in (-\pi, 0) \), then \( \sin(x) < 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f'(x) = 2\sin(x)(\cos(x) - 1) > 0 \]
and hence \( f \) is increasing on the interval \((-\pi, 0)\).
Also, if \( x \in (0, \pi) \), then \( \sin(x) > 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f'(x) = 2 \sin(x)(\cos(x) - 1) < 0 \]
and hence \( f \) is decreasing on the interval \((-\pi, 0)\).

We conclude that \( f \) has a local maximum value of 2 that occurs at \( x = 0 \).

Next, let us examine where the second derivative of \( f \) is equal to zero:

The solutions of the equation
\[ 2(2 \cos(x) + 1)(\cos(x) - 1) = 0. \]
occurs either where
\[ 2 \cos(x) + 1 = 0 \]
or where
\[ \cos(x) - 1 = 0. \]

We have already observed that the only solution of \( \cos(x) - 1 = 0 \) that lies in the interval \([-\pi, \pi]\) is \( x = 0 \). The equation \( 2 \cos(x) + 1 = 0 \) is equivalent to
\[ \cos(x) = -\frac{1}{2}. \]

The only solutions of this equation that lie in \([-\pi, \pi]\) are \( x = -2\pi/3 \) and \( x = 2\pi/3 \). These (and \( x = 0 \)) are possible inflection points of \( f \).

Now, note that if \( x \in (-\pi, -\frac{2\pi}{3}) \), then \( 2 \cos(x) + 1 < 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f''(x) = (2 \cos(x) + 1)(\cos(x) - 1) > 0 \]
and that \( f \) is concave up on the interval \((-\pi, -\frac{2\pi}{3})\).

If \( x \in \left(-\frac{2\pi}{3}, 0\right)\), then \( 2 \cos(x) + 1 > 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f''(x) = (2 \cos(x) + 1)(\cos(x) - 1) < 0 \]
and that \( f \) is concave down on the interval \((-\frac{2\pi}{3}, 0)\).

If \( x \in \left(0, \frac{2\pi}{3}\right)\), then \( 2 \cos(x) + 1 > 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f''(x) = (2 \cos(x) + 1)(\cos(x) - 1) < 0 \]
and that \( f \) is concave down on the interval \( \left(0, \frac{2\pi}{3}\right)\).

If \( x \in \left(\frac{2\pi}{3}, \pi\right)\), then \( 2 \cos(x) + 1 < 0 \) and \( \cos(x) - 1 < 0 \). This means that
\[ f''(x) = (2 \cos(x) + 1)(\cos(x) - 1) > 0 \]
and that \( f \) is concave up on the interval \( \left(\frac{2\pi}{3}, \pi\right)\).

We see that \( f \) has inflection points at \( x = -2\pi/3 \) and \( x = 2\pi/3 \).

Here is a summary of our results:

- \( f \) is increasing on the interval \((-\pi, 0)\).
- \( f \) has a local maximum value of 2 that occurs at \( x = 0 \).
- \( f \) is decreasing on the interval \((-\pi, 0)\).
- \( f \) is concave up on the intervals \((-\pi, -\frac{2\pi}{3})\) and \( \left(\frac{2\pi}{3}, \pi\right)\).
- \( f \) is concave down on the interval \((-\frac{2\pi}{3}, 0)\).
- \( f \) has inflection points at \( x = -2\pi/3 \) and \( x = 2\pi/3 \).

The graph of \( f \) is shown below.