Section 4.6 – Answers and Solutions to Selected Exercises

Problems 1, 2, 7, 9, 10, 13, 14, 19, 21, 23, 30, and 35.

1. We want to find two numbers whose sum is 23 and whose product is as large as possible.
   Let \( x \) be one of the numbers. Then the other number is \( 23 - x \) and the product of these numbers is \( x(23 - x) = 23x - x^2 \).
   We want to find where the maximum of the function \( P = 23x - x^2 \) occurs.
   Since \( P' = 23 - 2x \), we see that the critical numbers of \( P \) occur where \( 23 - 2x = 0 \).
   The solution of this equation is \( x = 11.5 \).
   Also, since \( P'' = -2 < 0 \), the critical number \( x = 11.5 \) must be a local minimum of \( P \).
   We conclude that the two numbers whose sum is 23 and whose product is as large as possible are 11.5 and 11.5.

10. We will build our box by starting with the flat piece of cardboard shown below, cutting along the perforations, and then folding. (The question really is where should we make the perforations.)

The amount of material used in making this box is \( x^2 + 4xy \).
However, we know that we want to have
\[ x^2y = 32,000 \text{ cm}^3. \]
The amount of material needed is thus
\[ A = x^2 + 4x \left( \frac{32,000}{x^2} \right) = x^2 + 128,000x^{-1}. \]
Since
\[ A' = 2x - 128,000x^{-2}, \]
we see that the critical points of \( A \) occur where
\[ 2x - \frac{128,000}{x^2} = 0. \]
We can write this equation as
\[ 2x^3 - 128,000 = 0 \]
or as
\[ x^3 = 64,000. \]
The solution of this equation is \( x = 40 \) cm. This critical number corresponds to
\[ y = \frac{32,000}{(40)^2} = 20 \text{ cm}. \]
Also note that
\[ A'' = 2 + 2(128,000)x^{-3}, \]
which is positive when \( x = 40 \)
so the critical number \( x = 40 \) corresponds to a local minimum of \( A \).
We conclude that the dimensions of our box should be \( 40 \text{ cm} \times 40 \text{ cm} \times 20 \text{ cm} \).

21. Here is the picture of the situation:
Assuming that a cut is indeed made, the total area enclosed by the square and the triangle is
\[ \left( \frac{1}{4}x \right)^2 + \frac{1}{2} \left( \frac{1}{3}y \right)h. \]
We would like to express the total area in terms of one variable only. We will get rid of the \( y \) and the \( h \): Clearly, \( y = 10 - x \).
Also,
\[ \sin(60^\circ) = \frac{h}{\frac{1}{3}y}, \]
which tells us that
\[ h = \frac{1}{3}y \sin(60^\circ) = \frac{\sqrt{3}}{6}y = \frac{\sqrt{3}}{6}(10 - x). \]
We now see that the total enclosed area (expressed in terms of \( x \) only) is
\[ A = \frac{1}{16}x^2 + \frac{1}{6}(10 - x) \left( \frac{\sqrt{3}}{6}(10 - x) \right) = \frac{1}{16}x^2 + \frac{\sqrt{3}}{36}(10 - x)^2. \]
Next, we compute
\[ A' = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10 - x) \]
\[ = \frac{1}{8}x - \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}}{18}x \]
\[ = \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} \]
\[ = \left( \frac{9 + 4\sqrt{3}}{72} \right)x - \frac{40\sqrt{3}}{72}. \]
The critical number of \( A \) occurs where
\[ \left( \frac{9 + 4\sqrt{3}}{72} \right)x - \frac{40\sqrt{3}}{72} = 0. \]
Solving this equation gives
\[ x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.35 \text{ m}. \]
Since
\[ A'' = \frac{9 + 4\sqrt{3}}{72} > 0, \]
we see that this critical number corresponds to a local minimum of \( A \). It corresponds to a total enclosed area of
\[ A_{x=4.35} = \frac{1}{16} (4.35)^2 + \frac{\sqrt{3}}{36}(10 - 4.35)^2 \approx 2.72 \text{ m}^2. \]
Now, we still must check two more things: What if we don’t cut the wire at all? If we don’t cut the wire at all and bend the whole thing into a square, then this
square has area

\[ A_{|x=10} = \frac{1}{16} (10)^2 = 6.25 \text{ m}^2. \]

If we don’t cut the wire at all and bend the whole thing into an equilateral triangle, then this triangle has area

\[ A_{|x=0} = \frac{\sqrt{3}}{36} (10 - 0)^2 \approx 4.81 \text{ m}^2. \]

Our conclusions are as follow:

In order to minimize the total enclosed area, we cut the wire such that \( x \approx 4.35 \text{ m} \) and we bend the piece of length 4.35 m into a square and bend the remaining piece into a triangle. This will give a total enclosed area of about 2.72 m².

In order to maximize the total enclosed area, we don’t cut the wire at all. We just bend the whole thing into a square. This will give an enclosed area of 6.25 m².

Here is the graph of the area function

\[ A = \frac{1}{16} x^2 + \frac{\sqrt{3}}{36} (10 - x)^2 \]

By looking at this graph, we can see that the absolute minimum of this function on the interval [0, 10] occurs at \( x \approx 4.35 \) and that the absolute maximum occurs at \( x = 10 \).