Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Circle True or False for each of the following statements.

(a) If a function \( f \) is continuous at a number \( a \), then \( \lim_{x \to a} f(x) \) must exist. (True)

(b) If \( \lim_{x \to a} f(x) \) exists, then the function \( f \) must be continuous at the number \( a \). (False)

(c) If the function \( f \) is continuous at the number 4 and if \( f(4) = 12 \), then it must be true that \( \lim_{x \to 4} f(x) = 12 \). (True)

(d) If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist, then it must be true that
\[
\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).
\]
(True)

(e) If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = \infty \), then it must be true that
\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right)
\]
does not exist. (False)

2. Consider the limit problem:
\[
\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x}.
\]

(a) Guess an approximate answer to the above problem by completing the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{\sqrt{3 + x} - \sqrt{3}}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.28631</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.29112</td>
</tr>
<tr>
<td>0.05</td>
<td>0.28748</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.28999</td>
</tr>
<tr>
<td>0.0023</td>
<td>0.28862</td>
</tr>
<tr>
<td>-0.0023</td>
<td>0.28873</td>
</tr>
</tbody>
</table>

Guess: Based on numerical calculations, the value of the above limit is approximately equal to _____ 0.28 __________.
(b) Use algebra to find the *exact* value of the limit in question. (Show your solution in detail.)

Since, for all \( x \neq 0 \) (and such that \( x \geq -3 \)), we have

\[
\frac{\sqrt{3 + x} - \sqrt{3}}{x} = \frac{\sqrt{3 + x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3 + x} + \sqrt{3}}{\sqrt{3 + x} + \sqrt{3}}
\]

\[
= \frac{(3 + x) - 3}{x(\sqrt{3 + x} + \sqrt{3})}
\]

\[
= \frac{1}{\sqrt{3 + x} + \sqrt{3}},
\]

then

\[
\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{3 + x} + \sqrt{3}}
\]

\[
= \frac{1}{2\sqrt{3}}
\]

\[
= \frac{\sqrt{3}}{6}.
\]

Thus, the exact value of the limit in question is \( \sqrt{3}/6 \). Note that the calculator–approximated value of this number is

\[
\frac{\sqrt{3}}{6} \approx 0.28868
\]

and this agrees with the computations that we did in part a.

3. For the function, \( f \), whose graph is given, state the value of the given quantity if it exists. If it does not exist, then explain why.
(a) $\lim_{x \to -2} f(x) = 0$
(b) $f(-2) = 1$
(c) $\lim_{x \to 1} f(x)$ does not exist because $\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = -2$.
(d) $f(1) = -2$

4. Match each function in (a)–(e) with one of the graphs (1)–(5). The correct matchings are (1,d), (2,b), (3,e), (4,c) and (5,a).
5. Use either the definition

\[ m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

or the definition

\[ m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

(whichever definition you prefer) to show that the parabola

\[ y = x^2 - 3x - 6 \]
has tangent line with slope \( m = -7 \) at the point \( P (-2, 4) \).

(Include all relevant details so that it is clear to the reader exactly how one or the other of the above definitions is being used.)

For all \( x \neq -2 \) we have

\[
\frac{f(x) - f(a)}{x - a} = \frac{f(x) - f(-2)}{x - (-2)}
\]
\[
= \frac{(x^2 - 3x - 6) - 4}{x + 2}
\]
\[
= \frac{x^2 - 3x - 10}{x + 2}
\]
\[
= \frac{(x + 2)(x - 5)}{x + 2}
\]
\[
= x - 5.
\]

Thus, the slope of the tangent line to the curve \( y = x^2 - 3x - 6 \) at the point \( P (-2, 4) \) is

\[
m = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} (x - 5) = -2 - 5 = -7.
\]