1. If \( g \) is the function \( g(x) = 2^x \), then

\[
g'(4) = ?
\]

Circle the correct choice (A–J) and give the reason for your choice. (Hint: If we are given the function \( g \) and a number \( a \), then how is \( g'(a) \) defined?)

Solution: The correct answer is J because

\[
g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}
\]

and in this case we have

\[
g'(4) = \lim_{x \to 4} \frac{g(x) - g(4)}{x - 4} = \lim_{x \to 4} \frac{2^x - 2^4}{x - 4}.
\]

2. The graph of a function \( f \) is shown below. One of the four graphs (A–D) is the graph of the derivative function of \( f \). Circle the correct choice and briefly explain the reasons for your choice.
Solution: The correct answer is A. There are many ways to see why. For example, there are six places on the graph of \( f \) where the tangent line to the graph is horizontal, and graph A has six corresponding points where the value of the function in the graph is zero. None of the other graphs (B, C, or D) has this feature.

3. The graph of the derivative, \( g' \), of a continuous function, \( g \), is shown below.
(a) The function $g$ is increasing on the intervals $(-3, 0)$ and $(3, \infty)$ and is decreasing on the intervals $(-\infty, -3)$ and $(0, 3)$.

(b) At what values of $x$ does $g$ have a local maximum? ($x = 0$). At what values of $x$ does $g$ have a local minimum? ($x = -3$ and $x = 3$).

(c) The graph of the function $g$ is concave up on the intervals $(-\infty, -1.8)$ and $(1.8, \infty)$ and is concave down on the interval $(-1.8, 1.8)$.

(d) At what values of $x$ does the graph of $g$ have an inflection point? ($x = -1.8$ and $x = 1.8$). (Note: The values $-1.8$ and $1.8$ in parts c and d are obtained by visual approximation. It is hard to tell the exact values by just looking at the graph.)

(e) Assuming that $g(0) = 0$, provide a sketch of the graph of $g$.

4. Find the derivatives of the following functions. You may use whatever differentiation
formulas and rules are necessary. Do not simplify your answers.

a) \( f(x) = -3x^4 + 3x^3 - 2x^2 + 2x - 1 \)

b) \( V(r) = \frac{4}{3}\pi r^3 \)

c) \( y = \sqrt{x} (x - 1) \)

d) \( y = e^x + \frac{4}{x} + \frac{3}{x^2} \)

\[ a) \quad f'(x) = -3 \left( 4x^3 \right) + 3 \left( 3x^2 \right) - 2 \left( 2x \right) + 2 \]
\[ = -12x^3 + 9x^2 - 4x + 2 \]

b) \( V'(r) = \frac{4}{3}\pi \left( 3r^2 \right) = 4\pi r^2 \)

c) \( \frac{dy}{dx} = \sqrt{x} \cdot \frac{d}{dx} (x - 1) + (x - 1) \frac{d}{dx} (\sqrt{x}) \)
\[ = \sqrt{x} + (x - 1) \cdot \frac{1}{2\sqrt{x}} \]

d) \( \frac{dy}{dx} = \frac{d}{dx} (e^x) + 4 \frac{d}{dx} (x^{-1}) + 3 \frac{d}{dx} (x^{-2}) \)
\[ = e^x - 4 \left( -x^{-2} \right) + 3 \left( -2x^{-3} \right) \]
\[ = e^x - \frac{4}{x^2} - \frac{6}{x^3} \]

5. Find the derivatives of the following functions. You may use whatever differentiation formulas and rules are necessary. Do not simplify your answers.

a) \( y = \frac{e^x}{1 + x} \)

b) \( f(x) = x^2e^x \)

c) \( h(x) = \frac{x + 2}{x - 1} \)

d) \( g(x) = \frac{x^3 - 2x^2 + 4x}{x} \)

(a) By the Quotient Rule

\[ \frac{dy}{dx} = \frac{(1 + x) \frac{d}{dx} (e^x) - e^x \frac{d}{dx} (1 + x)}{(1 + x)^2} \]
\[ = \frac{(1 + x) e^x - e^x}{(1 + x)^2} \]

(b) By the Product Rule

\[ f'(x) = x^2 \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x^2) \]
\[ = x^2 e^x + 2x e^x \]
(c) By the Quotient Rule

\[ h'(x) = \frac{(x - 1) \frac{d}{dx} (x + 2) - (x + 2) \frac{d}{dx} (x - 1)}{(x - 1)^2} \]
\[ = \frac{(x - 1) - (x + 2)}{(x - 1)^2} \]
\[ = \frac{-3}{(x - 1)^2} \]

(d) Since

\[ g(x) = \frac{x^3 - 2x^2 + 4x}{x} = x^2 - 2x + 4, \]

we have

\[ g'(x) = 2x - 2. \]

6. How many tangent lines to the curve \( y = x^2 \) pass through the point \((0, -12)\)? At which points to these tangent lines touch the curve \( y = x^2 \)? Show your solution in detail. Write in complete sentences that explain your reasoning process. Include all relevant calculations and also include a picture that illustrates this problem.

**Solution:** The tangent line to the curve \( y = x^2 \) at the point \((x, x^2)\) on the curve has slope \( dy/dx = 2x \). Also, the line that passes through the points \((x, x^2)\) and \((0, -12)\) has slope

\[ \frac{x^2 - (-12)}{x - 0} = \frac{x^2 + 12}{x}. \]

We want to find any values of \( x \) for which these slopes are the same. Solving the equation

\[ 2x = \frac{x^2 + 12}{x}, \]

we obtain \( x^2 = 12 \) or \( x = \pm \sqrt{12} \). Thus, there are two points on the curve \( y = x^2 \) where the tangent line to the curve at those points also passes through the point \((0, -12)\). The points are \((-\sqrt{12}, 12)\) and \((\sqrt{12}, 12)\).