Qualitative Properties of Functions

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By looking at the formula for a linear function,

\[ f(x) = mx + b, \]

we can quickly get a mental image of what the graph of \( f \) looks like. We know that the graph is a line that intersects the vertical axis at the point \((0, b)\) and the slope, \( m \), tells us whether \( f \) is increasing, decreasing, or constant. In particular:

1. If \( m > 0 \), then \( f \) is increasing.
2. If \( m < 0 \), then \( f \) is decreasing.
3. If \( m = 0 \), then \( f \) is constant.

Examples of increasing, decreasing, and constant linear functions are shown in Figures 1, 2, and 3. (Note that the function in Figure 1 is the function for converting Celsius temperature to Fahrenheit temperature.)

The words “increasing”, “decreasing”, and “constant” are words that tell us qualitative information about the behavior of a function. A qualitative description of a function is a description that tells us something about the overall behavior of the function in a broad sense but avoids referring to numerical detail. A description that does refer to numerical detail is called a quantitative description. As we will see, the terms “increasing”, “decreasing”, and some other qualitative terms are useful in describing the behavior of almost any function (not just linear functions). We will begin to explore this idea by looking at two examples.
Figure 1: $F = \frac{3}{5} C + 32$ is increasing.

Example 1 The graph of the function

$$g(x) = 4 - |x|$$

is shown in Figure 4.

The graph of $g$ consists of two line segments. The line segment that makes up the left half of the graph of $g$ has slope 1 and the line segment that makes up the right half of the graph of $g$ has slope -1. In fact, another way to write a formula for $g$ is to write it as a piecewise-defined function:

$$g(x) = \begin{cases} 
4 + x & \text{if } x < 0 \\
4 - x & \text{if } x \geq 0
\end{cases}.$$

Since the graph of $g$ consists of two line segments, we say that $g$ is a piecewise-linear function.

Now, suppose that we ask ourselves the question “Is $g$ increasing or decreasing?” Clearly, this question does not have a one word answer. The graph of $g$ is rising (going uphill) as $x$ ranges from $-\infty$ to 0 and falling (going downhill) as $x$ ranges from 0 to $\infty$. Hence, we can answer the question by saying that $g$ is increasing on the interval $(-\infty, 0)$ and decreasing on the interval $(0, \infty)$. Since $g$ is a piecewise-linear function, we can even be a little more specific by saying that $g$ is increasing with slope 1 on the interval $(-\infty, 0)$ and decreasing with slope -1 on the interval $(0, \infty)$. 

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Figure 2: $f(x) = -4x + 6$ is decreasing.

**Example 2** The graph of the function

$$h(x) = x^2 - 4x + 2$$

is shown in Figure 5.

Since $h$ is a quadratic function, its graph is a parabola. This parabola has vertex at the point $(2, -2)$.

How can we describe the increasing/decreasing behavior of the function $h$? Since the graph of $h$ is falling as $x$ ranges from $-\infty$ to $2$ and rising as $x$ ranges from $2$ to $\infty$, we can say that $h$ is decreasing on the interval $(-\infty, 2)$ and increasing on the interval $(2, \infty)$. Note that we cannot be as specific about the increasing/decreasing behavior of $h$ as we were able to be with the function $g$ in Example 1. In particular, the graph of $h$ is not made up of line segments (like the graph of $g$ is) so we can’t make a statement of the form “The graph of $h$ is decreasing with slope ___ on the interval $(-\infty, 2)$ and increasing with slope ___ on the interval $(2, \infty)$”. In fact, the “slope” of the graph of $h$ appears to be different at each different point on the graph.

1 Increasing/Decreasing Behavior of Functions

Having studied linear functions and the nonlinear functions $g$ and $h$ in Examples 1 and 2, we now have a pretty good idea about how the words “increasing” and “decreasing” are used to describe the behavior of functions. In particular, we realize that it is not always possible to label a function strictly
as “increasing” or “decreasing”. Instead, as with the functions $g$ and $h$ in Examples 1 and 2, we must usually make reference to the increasing/decreasing behavior of a function over various intervals that make up the domain of the function. The formal definitions of the terms “increasing” and “decreasing” are as follows:

**Definition 3** Let $f$ be a function whose domain includes the interval $(a, b)$.

1. We say that the function $f$ is increasing on the interval $(a, b)$ if for every pair of values $x_1$ and $x_2$ in $(a, b)$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

2. We say that the function $f$ is decreasing on the interval $(a, b)$ if for every pair of values $x_1$ and $x_2$ in $(a, b)$ with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

To see that Definition 3 captures the essence of what we want the terms “increasing” and “decreasing” to mean, we illustrate the definition using some examples that we have already seen.

1. The function $f(x) = -4x + 6$ is decreasing on the interval $(-\infty, \infty)$ because if we take any two values $x_1$ and $x_2$ in $(-\infty, \infty)$ with $x_1 < x_2$, then we have $f(x_1) > f(x_2)$. For example, for $x_1 = 2$ and $x_2 = 8$, we have $f(x_1) = -2$ and $f(x_2) = -26$. 

Figure 3: $y = 5$ is constant.
2. The function \( g(x) = 4 - |x| \) is increasing on the interval \((-\infty, 0)\) because if we take any two values \( x_1 \) and \( x_2 \) in \((-\infty, 0)\) with \( x_1 < x_2 \), then we have \( g(x_1) < g(x_2) \). For example, for \( x_1 = -6 \) and \( x_2 = -2.5 \), we have \( g(x_1) = -2 \) and \( g(x_2) = 1.5 \).

3. The function \( h(x) = x^2 - 4x + 2 \) is increasing on the interval \((2, \infty)\) because if we take any two values \( x_1 \) and \( x_2 \) in \((2, \infty)\) with \( x_1 < x_2 \), then we have \( h(x_1) < h(x_2) \). For example, for \( x_1 = 4 \) and \( x_2 = 6.2 \), we have \( h(x_1) = 2 \) and \( h(x_2) = 15.64 \).

Graphically, the fact that a function \( f \) is increasing on an interval \((a, b)\) means that the graph of \( f \) is rising (or going “uphill”) from left to right over the interval \((a, b)\). Likewise, the fact that \( f \) is decreasing on an interval \((a, b)\) means that the graph of \( f \) is falling (or going “downhill”) from left to right over the interval \((a, b)\).

An obvious question now arises: If we are given a function \( f \), how can we determine on which intervals \( f \) is increasing and on which intervals \( f \) is decreasing? After we have developed the tools of Calculus (especially the concept of the derivative), we will be equipped with analytic techniques that will allow us to address this question in a systematic way. For the time being though, we will address the question as best as we can by drawing the graphs of functions by hand or by using a graphing calculator or graphing computer software.
Figure 5: Graph of $h(x) = x^2 - 4x + 2$

**Example 4** On which intervals is the function

$$p(x) = x^3 - 12x$$

increasing? On which intervals is $p$ decreasing?

**Solution:** A computer-generated graph of $p$ is shown in Figure 6.

Figure 6: Graph of $p(x) = x^3 - 12x$

By looking at Figure 6, it appears that $p$ is increasing on the interval $(-\infty, -2)$, $p$ is decreasing on the interval $(-2, 2)$, and $p$ is increasing on $(2, \infty)$. Since it is sometimes hard to interpret computer (or calculator) –
generated graphs with the human eye, we may be a little unsure as to whether
$x = -2$ and $x = 2$ are the exact points where the graph changes from increasing
to decreasing. For example, how can we be sure that it is not really the
case that $p$ is decreasing on the interval $(-2.1, 2.2)$ rather than on the interval
$(-2, 2)$ as we stated? The graph is not good enough and our eyes are not
strong enough to determine this with absolute certainty. The Calculus tools
that we are going to develop, however, will allow us to answer such questions
with certainty.

**Exercise 5** Graph each of the following functions (using a graphing device if
necessary) and do your best (based on your graph) to determine the intervals
on which each function is increasing and the intervals on which each function
is decreasing. Write your answers in complete sentences using the correct ter-
iminology. For example: “$f$ is increasing on the interval (____, ____).”

1. $f(x) = 2x + 2$
2. $f(x) = |x|$
3. $f(x) = -2x^2 - 16x - 25$
4. $f(x) = x^3 - 3x^2 - 9x$
5. $f(x) = 3x^4 - 4x^3 - 12x^3 + 5$ (HINT: When graphing this function,
   set your viewing rectangle to $X_{\text{min}} = -2, X_{\text{max}} = 3, Y_{\text{min}} = -30,$
   $Y_{\text{max}} = 20.$)
6. $f(x) = x^4 - 4x^3$ (HINT: When graphing this function, set your viewing
   rectangle to $X_{\text{min}} = -5, X_{\text{max}} = 5, Y_{\text{min}} = -30, Y_{\text{max}} = 10.$)

2 Relative Maximum and Minimum Values

One of the applications of Calculus is to solve problems that involve finding
relative maximum and/or relative minimum values of a function. Such
problems, which are called optimization problems, arise when we are try-
ing to determine the “biggest” or “smallest” value that some quantity can
have. For example, we might be trying to solve a problem in which we want
to maximize profit, or we might be trying to solve a problem in which we
want to minimize the time it takes to do a certain job. The terms relative
maximum and relative minimum are defined as follows:
Definition 6 Let $f$ be a function whose domain contains the point $x_0$ and suppose that $f (x_0) = y_0$.

1. $f$ is said to have a relative maximum value of $y_0$ occurring at $x_0$ if there exists an interval $(a, b)$, contained in the domain of $f$ and containing $x_0$, such that $f (x_0) \geq f (x)$ for all $x$ in $(a, b)$.

2. $f$ is said to have a relative minimum value of $y_0$ occurring at $x_0$ if there exists an interval $(a, b)$, contained in the domain of $f$ and containing $x_0$, such that $f (x_0) \leq f (x)$ for all $x$ in $(a, b)$.

Graphically, a relative maximum value of a function corresponds to a point, $(x_0, y_0)$, on the graph of the function that is higher than all other nearby points on the graph. Likewise, a relative minimum value of a function corresponds to a point, $(x_0, y_0)$, on the graph of the function that is lower than all other nearby points on the graph. In either case, we say that $f$ has a relative extreme value of $y_0$ occurring at $x_0$.

Example 7 The function $g (x) = 4 - |x|$ has a relative maximum value of 4 occurring at 0. This can easily be seen by looking at the graph of $g$ in Figure 4. The point $(0, 4)$ is the highest point on the graph relative to other points around it. In fact, the point $(0, 4)$ is the highest point on the graph!

Example 8 The function $h (x) = x^2 - 4x + 2$ has a relative minimum value of $-2$ occurring at 2. This can easily be seen by looking at the graph of $h$ in Figure 5. The point $(2, -2)$ is the lowest point on this graph.

Example 9 The function $f (x) = -4x + 6$ does not have any relative maximum or relative minimum values. Since the graph of $f$ is a line, there are no points on the graph that are higher or lower than all nearby points on the graph.

Example 10 According to our graphical analysis in Example 4, it appears that the function $p(x) = x^3 - 12x$ (whose graph is shown in Figure 6) has a relative maximum occurring at $-2$ and a relative minimum occurring at 2. Assuming that our graphical analysis is correct, we can determine what the corresponding relative maximum and minimum values of $p$ are by simply evaluating

$$p(-2) = (-2)^3 - 12(-2) = 16$$
and

\[ p(2) = (2)^3 - 12(2) = -16. \]

Thus, \( p \) has a relative maximum value of 16 occurring at \(-2\) and a relative minimum value of \(-16\) occurring at \(2\). This example is a good illustration of why we include the word “relative” in our terminology. In particular, note that the point \((-2, 16)\) is not actually the highest point on the graph of \( p \) but it is the highest point on the graph relative to all other nearby points on the graph. The same type of comment applies to the point \((2, -16)\), which is not the lowest point on the graph but which is the lowest relative to points around it.

**Exercise 11** Determine the relative maximum and relative minimum values (and where they occur) for each of the functions that you graphed in Exercise Set 5. Write your answers in complete sentences using the correct terminology. For example: “\( f \) has a relative maximum value of \( \ldots \) occurring at \( \ldots \).”

**Exercise 12**

1. Draw (by hand) the graph of a function, \( f \), that is increasing on the interval \(( -\infty, 4 )\) and decreasing on the interval \(( 4, \infty )\).

2. Draw (by hand) the graph of a function, \( f \), that is decreasing on the interval \(( -\infty, -3 )\), constant on the interval \(( -3, 1 )\), and decreasing on the interval \(( 1, \infty )\).

3. Draw the graph of a function, \( f \), that satisfies both of the following properties:

   - \( f(x) > 0 \) for all \( x \) in \(( -\infty, \infty )\).
   - \( f \) is increasing on \(( -\infty, \infty )\).

   **Thinking back to the functions you studied in your Precalculus course, can you think of an actual function \( f \) that fits this description?**

4. **Change the following statement in order to make it be correct:** If a function \( f \) is decreasing on the interval \(( -\infty, -4 )\) and increasing on the interval \(( -4, \infty )\), then \( f \) has a relative maximum value at \( x = -4 \).
5. Suppose that \( f \) is a function whose domain includes the interval \((0, 10)\). Which of the following assertions must be true. Explain your answers.

(a) If \( f(2) < f(8) \), then \( f \) is increasing on \((0, 10)\).

(b) If \( f \) is increasing on \((0, 10)\), then \( f(2) < f(8) \).

(c) If \( f \) is increasing on \((2.86, 3)\) and decreasing on \((3, 3.6)\), then \( f \) has a relative maximum occurring at 3.

(d) If \( f \) has a relative maximum occurring at 3, then \( f \) is increasing on \((2.86, 3)\) and decreasing on \((3, 3.6)\).

(e) If \( f(4) = -2 \) and \( f(x) \geq -2 \) for all \( x \) in \((3.7, 4.1)\), then \( f \) has a relative minimum occurring at 4.