Using Derivatives in Problem Solving

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The derivative is one of the most prominent and powerful tools that we can use in trying to understand the behavior of functions. In fact, the concept of “derivative” is the nonlinear analog of the linear concept of “slope”. Just as the slope, \( m \), of a linear function, \( y = mx + b \) tells us the rate of change of \( y \) as \( x \) increases, the derivative, \( f' \), of a nonlinear function, \( y = f(x) \), also tells us the rate of change of \( y \) as \( x \) increases. Of course, the difference is that the rate of change is constant for linear functions but changes from point to point for nonlinear functions.

The fact that \( f' \) tells us the slopes of tangent lines to the graph of \( f \) can be used in solving problems that require a knowledge of intervals over which \( f \) is increasing and intervals over which \( f \) is decreasing. Such problems include optimization problems in which we are trying to determine the biggest or smallest value that \( f \) achieves. The fact that \( f' \) tells us the rate of change of \( f \) can be used in solving rate problems - problems in which we want to know how fast some process (described by the function \( f \)) is occurring at some particular instant (or at all instants).

1 The Derivative as Slope

One of the most important pieces of information given to us by the derivative, \( f' \), of a function, \( f \), is information about the increasing/decreasing behavior of \( f \). Consequently, the derivative can be used to determine the locations of the relative maxima and minima of \( f \). Intervals where \( f'(x) > 0 \) are intervals on which the tangent line to the graph of \( f \) has positive slope at each point. Consequently, \( f \) is increasing on these intervals. Likewise, intervals where \( f'(x) < 0 \) are intervals on which \( f \) is decreasing.
As an example, if we take the function \( f(x) = x^2 \), the derivative of \( f \) is 
\[ f'(x) = 2x \]
and it is easy to see that \( 2x < 0 \) for all \( x \) in the interval \((-\infty, 0)\) and \( 2x > 0 \) for all \( x \) in the interval \((0, \infty)\). This tells us that \( f \) is decreasing on the interval \((-\infty, 0)\) and increasing on the interval \((0, \infty)\). (See Figures 1 and 2.) Because \( f \) is decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\), we know that \( f \) achieves a local minimum at \( x = 0 \).

![Figure 1: Graph of \( f(x) = x^2 \)](image1)

![Figure 2: Graph of \( f'(x) = 2x \)](image2)

The general idea illustrated by the above simple example is described in Theorem 1.
Theorem 1 Suppose that $f$ is a differentiable function whose domain includes the open interval $(a,b)$ (where, possibly, $a = -\infty$ and/or $b = \infty$).

If $f'(x) > 0$ for all $x$ in $(a,b)$, then $f$ is increasing on $(a,b)$, and if $f'(x) < 0$ for all $x$ in $(a,b)$, then $f$ is decreasing on $(a,b)$.

If a differentiable function, $f$, is increasing on an interval $(a,b)$ and decreasing on an interval $(b,c)$, then it must be the case that $f'(b) = 0$ and that $f$ achieves a local maximum value at $x = b$. Likewise, if $f$ is decreasing on $(a,b)$ and increasing on $(b,c)$, then it must be the case that $f'(b) = 0$ and that $f$ achieves a local minimum value at $x = b$. This idea is stated formally in Corollary 2.

Corollary 2 Suppose that $f$ is a differentiable function whose domain includes the open interval $(a,c)$ (where, possibly, $a = -\infty$ and/or $c = \infty$) and suppose that $b$ is a point in $(a,c)$.

If $f'(x) > 0$ for all $x$ in $(a,b)$ and $f'(x) < 0$ for all $x$ in $(b,c)$, then $f$ achieves a local maximum value at $x = b$.

If $f'(x) < 0$ for all $x$ in $(a,b)$ and $f'(x) > 0$ for all $x$ in $(b,c)$, then $f$ achieves a local minimum value at $x = b$.

Since the proofs of Theorem 1 and Corollary 2 require the Mean Value Theorem, we will give the proofs of the theorem and corollary after we have studied the Mean Value Theorem. However, we note that Theorem 1 and Corollary 2 are “intuitively obvious” when we think of the derivative of $f$ at any point as the slope of the tangent line to the graph of $f$ at that point. It agrees with intuition, for example, that if a function, $f$, has a tangent with positive slope at each point in some interval $(a,b)$, then $f$ should be increasing throughout $(a,b)$.

Example 3 In this example, we use the derivative of the function

$$f(x) = x^3 - 12x$$

to determine the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing. We also find the local maximum and minimum values of $f$ and where they occur.

The graph of $f$ is shown in Figure 3. Just by looking at this graph, it appears that $f$ is increases on the interval $(-\infty, -2)$, achieves a local maximum at $x = -2$, decreases on the interval $(-2, 2)$, achieves a local minimum at
Figure 3: Graph of \( f(x) = x^3 - 12x \)

\( x = 2 \), and increases on the interval \((2, \infty)\). We will use the derivative of \( f \) to verify that this graphical observation is indeed correct. (Recall that graphs drawn by machines are not perfect, and our eyes can fool us too. Might it be that \( f \) really achieves a local maximum at \( x = -2.1 \) instead of at \( x = -2 \)?)

The derivative of \( f \) is

\[
f'(x) = 3x^2 - 12
\]

and we must determine the intervals on which \( f'(x) > 0 \) and the intervals on which \( f'(x) < 0 \). Since \( f' \) is itself a polynomial function, and hence continuous, the most efficient way to do this is to determine where \( f'(x) = 0 \). To find where \( f'(x) = 0 \), we must solve the equation

\[
3x^2 - 12 = 0.
\]

The solutions of this equation are easily seen to be \( x = -2 \) and \( x = 2 \). We conclude that \( f'(-2) = 0 \) and \( f'(2) = 0 \). What remains is to determine the sign (positive or negative) of \( f' \) on the intervals \((-\infty, -2)\), \((-2, 2)\), and \((2, \infty)\). One way to do this is to graph \( f' \) (Figure 4). By looking at the graph of \( f' \), it can be seen that \( f'(x) > 0 \) on the intervals \((-\infty, -2)\) and \((2, \infty)\) and that \( f'(x) < 0 \) on the interval \((-2, 2)\).

An alternative (algebraic) way to determine the sign of \( f' \) is to write the formula for \( f' \) in factored form as

\[
f'(x) = 3(x^2 - 4)
\]

(1)
from which we can see that the sign of \( f' \) depends on the sign of the factor \( x^2 - 4 \). If \( x \) is in the interval \((-\infty, -2)\) or in the interval \((2, \infty)\), then \( x^2 > 4 \) which means that \( x^2 - 4 > 0 \). If \( x \) is in the interval \((-2, 2)\), then \( x^2 < 4 \) which means that \( x^2 - 4 < 0 \). This algebraic reasoning shows us that \( f'(x) > 0 \) at all points, \( x \), in the intervals \((-\infty, -2)\) and \((2, \infty)\), and \( f'(x) < 0 \) at all points, \( x \), in the interval \((-2, 2)\). Our conclusion is that \( f \) is increasing on the interval \((-\infty, -2)\), decreasing on the interval \((-2, 2)\), and increasing on the interval \((2, \infty)\). By applying Corollary 2, we conclude that \( f \) achieves a local maximum value at \( x = -2 \) and a local minimum value at \( x = 2 \). The local maximum and local minimum values of \( f \) are, respectively,

\[
  f(-2) = (-2)^3 - 12(-2) = 16
\]

and

\[
  f(2) = (2)^3 - 12(2) = -16.
\]

### 1.0.1 Exercises

In a previous homework exercise, you determined intervals of increase/decrease and relative maximum/minimum values of the functions, \( f \), given in 1-5 below. You did this by simply graphing these functions and looking at the graphs. Check the accuracy of your previous results by using the derivative, \( f' \), of each function \( f \). (Since each function is a polynomial, you should know how to compute its derivative!) Also, for each function, \( f \), show a graph of
\( f \) and a graph of \( f' \) and comment on how the relationship between the two graphs “makes sense”.

1. \( f(x) = 2x + 2 \)
2. \( f(x) = -2x^2 - 16x - 25 \)
3. \( f(x) = x^3 - 3x^2 - 9x \)
4. \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \)
5. \( f(x) = x^4 - 4x^3 \)

### 1.1 Optimization Problems

An optimization problem is a problem in which we want to find the biggest or the smallest that some quantity can be. If the quantity to be optimized is described by a differentiable function, \( f \), then we can use the derivative of \( f \) to determine where \( f \) achieves its relative maximum and minimum values, and hence find a solution to our optimization problem. Of course, to solve an optimization problem in this way requires that we know how to differentiate the function \( f \). At this point, we only know how to differentiate polynomial functions, so the examples of optimization problems that we have selected to show here are ones involving only polynomials. We will continue to revisit optimization problems as we continue to develop more differentiation formulas (for trigonometric functions, exponential functions, etc.).

**Example 4** Suppose that we want to construct a rectangle that has perimeter 100 ft. and whose area is as large as possible.

The first thing to realize in trying to solve this problem is that there are many possible ways to construct a rectangle with perimeter 100 ft, but that different constructions give different areas. For example, if we construct a rectangle with sides 30, 20, 30, 20 (Figure 5), then this rectangle has perimeter 100 ft and area

\[
(30 \text{ ft})(20 \text{ ft}) = 600 \text{ ft}^2.
\]

If we construct a rectangle with sides 35, 15, 35, 15 (Figure 6), then this rectangle has perimeter 100 ft and area

\[
(35 \text{ ft})(15 \text{ ft}) = 525 \text{ ft}^2.
\]
Figure 5: A $30 \times 20$ Rectangle

Figure 6: A $35 \times 15$ Rectangle
Figure 7: An $(x) \times (50 - x)$ Rectangle

In general, if we define $x$ to be the length of one of the sides of our rectangle (Figure 7), then the rectangle has sides of length $x$, $50 - x$, $x$, and $50 - x$ and the area of the rectangle is

$$A = x(50 - x) = 50x - x^2.$$  

In this problem, since $x$ stands for the length of one of the sides of the rectangle, the only admissible (pertinent) values of $x$ are values between 0 and 50. Thus, the problem that we want to solve is to find the maximum value of the function

$$f(x) = 50x - x^2$$

on the interval $[0, 50]$.

Since $f$ is a quadratic function, it is easy to draw the graph of $f$ (Figure 8) and to see that the maximum value of $f$ occurs at $x = 25$. Hence, Calculus is really not necessary to solve this problem.

Nonetheless, we can use Calculus to complete the solution by computing the derivative

$$f'(x) = 50 - 2x$$

and observing that $f'(x) = 0$ when $x = 25$. Furthermore, $f'(x) > 0$ for all $x$ in the interval $(0, 25)$ and $f'(x) < 0$ for all $x$ in the interval $(25, 50)$. This means that $f$ has a relative (and actually absolute) maximum value occurring at $x = 25$. Our conclusion is that the rectangle of maximum possible area
(and with perimeter 100 ft) is a square. The maximum possible area achieved is

\[(25 \text{ ft})(25 \text{ ft}) = 625 \text{ ft}^2.\]

**Example 5** A rectangular box (without a top) is to be constructed from a 6 ft \times 4 ft rectangular cardboard sheet by cutting equal size squares from each corner of the sheet, discarding these squares, and then folding the remaining cardboard into the shape of a box. (See Figure 9.) What size squares should be cut from the corners of the sheet in order to construct the box of greatest possible volume?

Solving this problem involves three steps:

1. Express the volume of the box, \(V\), as a function of the length of a side, \(x\), of a square to be cut from the corner of the cardboard sheet. This will give us a function of the form \(V = f(x)\).

2. Determine the relevant domain of \(f\).

3. Use the derivative, \(f'\), to find the absolute maximum value of \(f\) on the relevant domain.

The volume of a rectangular box is

\[\text{area of base} \times \text{height}.\]
The box we want to construct will have height $x$ (refer to Figure 9) and rectangular base with area $(6 - 2x)(4 - 2x)$. The volume of the box will thus be

$$V = f(x) = (6 - 2x)(4 - 2x)x.$$

Writing $f$ in expanded form, we obtain

$$f(x) = 4x^3 - 20x^2 + 24x.$$

A graph of $f$ is shown in Figure 10.

Since the greatest possible length of the side of a square that can be cut from the sheet to construct the box is 2 feet, we are really only interested in values of $x$ between 0 and 2. Thus the “relevant domain” of $f$ for this problem is the interval $[0, 2]$. The graph of $f$ on this domain is shown in Figure 11.

By looking at the graph in Figure 11, it is clear that $f$ achieves a single absolute maximum on the interval $[0, 2]$. To find where this maximum occurs, we use the derivative

$$f'(x) = 12x^2 - 40x + 24.$$
Figure 10: Graph of \( V = f(x) = 4x^3 - 20x^2 + 24x \)

Figure 11: Graph of \( V = (6 - 2x)(4 - 2x)x \)
The maximum we are looking for occurs where \( f'(x) = 0 \). Solving the equation

\[ 12x^2 - 40x + 24 = 0 \]

(by using the Quadratic Formula), we obtain the solutions

\[ x = \frac{5}{3} + \frac{1}{3}\sqrt{7} \approx 2.5486 \]

and

\[ x = \frac{5}{3} - \frac{1}{3}\sqrt{7} \approx 0.78475. \]

Only the second of these solutions \((x = 0.78475)\) is in the interval \([0, 2]\) and this is the one that corresponds to the absolute maximum of \(f\) on \([0, 2]\). (Look at Figure 10 to see why \(x = 2.5486\) is also a point at which \(f'(x) = 0\).

Our conclusion is that, in order to construct the box of greatest possible volume, we should cut squares of side length 0.78475 ft from each corner of the sheet. The resulting box will have volume

\[
(6 - 2(0.78475))(4 - 2(0.78475))(0.78475) = 8.4504 \text{ ft}^3.
\]

Examples 5 and 4 illustrate the basic plan of attack for solving optimization problems. The strategy can be outlined as follows:

**Strategy for Solving Optimization Problems**

1. Determine what quantity, \(Q\), is to be maximized or minimized and express \(Q\) as a function of some independent variable, \(x\). That is write a function, \(Q = f(x)\).

2. Determine the relevant domain of \(Q\). The relevant domain, which is usually an interval \([a, b]\), is determined by deciding which values of \(x\) are admissible in the particular problem being addressed.

3. Use the derivative, \(f'\), to determine the absolute maximum or minimum value of \(f\) on the relevant domain \([a, b]\).
1.1.1 Exercises

1. For the problem in Example 5, compute the volume of the box to be constructed using some other choices of \( x \) in \([0, 2]\) (other than the optimal value of \( x = 0.78475 \)). Whatever you choose, you should see that you will get a box with volume less than 8.4504 ft\(^3\).

2. Rework Example 5 under the assumption that the cardboard sheet that you are starting with has dimension 8 ft \( \times \) 8 ft. What is the box of largest volume you can construct in this case and how should you go about constructing it? (What size squares should be cut from the corners?)

3. Generalize Example 5 by reworking the example under the assumption that the cardboard sheet that you are starting with has dimension \( a \) ft \( \times \) \( b \) ft (where \( a \) and \( b \) are just assumed to be positive numbers with \( a \geq b \)). What is the box of largest volume you can construct in this case and how should you go about constructing it?

4. A farmer has 1000 ft of fence and he wants to construct a rectangular pen using all of this fence. What is the largest pen (in area) that the farmer can construct and what are the dimensions of this pen?

5. A farmer has 1000 ft of fence and he wants to construct a rectangular pen using all of this fence. Part of the fence will be used to put a divider in the middle of the pen so that pigs can be kept in one half and cows in the other half. What is the largest pen (in area) that the farmer can construct and what are the dimensions of this pen?

6. Find two numbers whose difference is 12 and whose product is a minimum. Hint: Let \( x \) be one of the numbers. Then the other number is \( x - 12 \).

7. Find the point on the line \( y = -3x + 5 \) that is closest to the point \((0, 0)\). Hint: The distance between two points in the plane, \((x_1, y_1)\) and \((x_2, y_2)\), is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

Also, a distance is minimized when its square is minimized.

8. Find the point on the line \( y = 2x \) that is closest to the point \((3, 0)\).
9. Find the points on the parabola \( y = x^2 \) that are closest to the point (0, 1).

10. A parabola is a curve consisting of points, \((x, y)\), that satisfy an equation of the form \( y = ax^2 + bx + c \) where \( a \), \( b \), and \( c \) are constants with \( a \neq 0 \). Use Calculus to explain why the vertex of a parabola occurs where \( x = -b/2a \).

## 2 The Derivative as Rate of Change

For a differentiable function, \( y = f(x) \), the derivative \( f'(x_0) \) is a measurement of the rate of change of \( f \) at \( x_0 \). It is for this reason that the derivative is useful in studying many types of problems in which we want to determine “how fast” some process is occurring.

### 2.1 Units of the Derivative and the Leibniz Notation

An example in which a derivative is interpreted as a rate of change is the example of a cone–shaped tank being filled with water (Example 3 beginning on page 9 of the handout “Derivatives”). In that example, the depth of the water in the tank (in feet) at time \( t \) minutes is given by a function

\[
D = f(t) = kt^{1/3} \quad \text{(where \( k \) is a constant)}.
\]

Since the function \( f \) gives water depth \( (D) \) as a function of time \( (t) \), the derivative of \( f \) tells us the rate of change of depth with respect to time. The rate of change of depth at any particular instant of time, \( t_0 \), is \( f'(t_0) \) feet/minute.

Note that the units of the derivative, \( f' \), in the above example, are feet/minute. In general, if \( y = f(x) \) is any differentiable function, then the units of the derivative, \( f' \), are

\[
\frac{\text{units of} \ y}{\text{units of} \ x}
\]

which can also be thought of as

\[
\frac{\text{units of the dependent variable}}{\text{units of the independent variable}}
\]
To see the reason for this, think about how the derivative is defined: The units of the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

are

$$\frac{\text{units of } y}{\text{units of } x}$$

and taking the limit of the difference quotient as $x \to x_0$ does not change these units so we have

$$\text{units of } f'(x_0) = \text{units of } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{\text{units of } y}{\text{units of } x}.$$ 

**Example 6** The diameter, $D$, of a circle of radius $r$ inches is

$$D = 2r \text{ inches}.$$ 

Hence, the diameter is a function,

$$f(r) = 2r,$$

of the radius. Since the independent variable ($r$) of $f$ has units of inches and the dependent variable ($D$) of $f$ also has units of inches, the derivative of $f$ has units of “inches per inch” (or in/in). These units make sense because the derivative,

$$f'(r) = 2$$

tells us

$$\frac{\text{change in diameter}}{\text{change in radius}}$$

and if we picture our circle to be “growing” (meaning that $r$ is increasing), then it is clear that the diameter ($D$) increases by two inches for each one inch increase in the radius ($r$).

We now introduce a notation for derivatives, called the *Leibniz notation*, which is commonly used in applications of Calculus that involve “rate of change” problems. Thus far, we have been using the notation $f'$ (read “f prime”) to denote the derivative (function) of a differentiable function $f$. 

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The “prime” notation was first used by the French mathematician Joseph–Louis LaGrange (1736-1813) but this was not the original notation for the derivative.

The derivative was originally invented (or should we say “discovered”?) during the late 1600s by Isaac Newton (1642-1727) of England and Gottfried Wilhelm von Leibniz\(^1\) (1646-1716) of Germany, neither of whom used the notation \( f' \). In fact, since they were working independently of each other, Newton and Leibniz both used different notations. To denote the derivative function of a differentiable function, \( y = f(x) \), Newton used the notation \( \dot{y} \) (now called the “dot” notation) and Leibniz used the notation \( dy/dx \). Both of these notations are still in use today along with the LaGrange “prime” notation. In the teaching of modern Calculus courses, the LaGrange notation is usually introduced first (as we have done) because this notation is more “formally correct” than either the Newton or the Leibniz notation. The reason that the Leibniz notation (and, to a lesser extent, the Newton notation\(^2\)) is still widely used is that it reminds us that \( dy/dx \) is the instantaneous rate of change of \( y \) with respect to \( x \). This is especially appealing in applied problems which focus on studying the rate of change of one variable with respect to another. In what follows, we focus our attention on the Leibniz notation, \( dy/dx \), and compare it with the LaGrange notation, \( f' \).

To see why Leibniz chose to use the \( dy/dx \) notation to denote the derivative function of a differentiable function \( y = f(x) \), let us return once again to the simple case of a linear function, \( y = mx + b \). The slope, \( m \), of this linear function can be computed by choosing any two points, \((x_0, y_0)\) and \((x, y)\) on the line (with \( x \neq x_0 \)). Having chosen two such points, if we define \( \Delta x = x - x_0 \) and \( \Delta y = y - y_0 \), then the slope of the line is

\[
m = \frac{\Delta y}{\Delta x}.
\]

The process of finding the “slope” (really the derivative) of a nonlinear function, \( y = f(x) \) at a point, \( x_0 \), is similar: Assuming that \( x \neq x_0 \), we define

\(^1\)Actually, the seminal ideas were introduced even before the time of Newton and Leibniz in the work of Fermat and others.

\(^2\)The Newton “dot” notation is used almost exclusively in applications where the independent variable is time, \( t \). Thus, for \( y = f(t) \), we would write \( \dot{y} \) to denote the derivative of \( y \) with respect to \( t \).
\[ \Delta x = x - x_0 \text{ and } \Delta y = f(x) - f(x_0) \] and consider the difference quotient, 

\[ \frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}. \]

Assuming that \( f \) is differentiable at \( x_0 \), taking the limit of the above difference quotient as \( x \to x_0 \) gives us 

\[ f'(x_0) = \lim_{x \to x_0} \frac{\Delta y}{\Delta x}. \]

The linear function, \( L \), that best approximates \( f \) near \( x_0 \) is 

\[ L(x) = f'(x_0)(x - x_0) + f(x_0). \]

The graph of \( L \) is what we call the tangent line to the graph of \( f \) at the point \((x_0, f(x_0))\). If we define \( dx = x - x_0 \) (for any \( x \neq x_0 \)) and \( dy = L(x) - f(x_0) \), we can write the equation of this tangent line as 

\[ dy = f'(x_0) \, dx \]

which is the same as 

\[ f'(x_0) = \frac{dy}{dx}. \]

Because Leibniz thought of the derivative in terms of \( dy \) and \( dx \) (as described above), he defined the notation \( dy/dx \) to mean 

\[ \frac{dy}{dx} = \lim_{x \to x_0} \frac{\Delta y}{\Delta x}. \quad (2) \]

An unpleasant problem that arises in defining \( dy/dx \) by equation (2) is that the left hand side (which is the notation being defined) makes no reference to the point \( x_0 \) at which the limit on the right is being computed. In fact, \( dy/dx \) as defined by equation (2) is a real number that depends on the choice of \( x_0 \). Thus, it is actually more correct to write 

\[ \frac{dy}{dx}(x_0) = \lim_{x \to x_0} \frac{\Delta y}{\Delta x} \]

or 

\[ \frac{dy}{dx} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta y}{\Delta x}. \quad (3) \]
The vertical bar with subscript \( x = x_0 \) emphasizes that we are evaluating the derivative at \( x = x_0 \). Hence, we adopt the convention that \( \frac{dy}{dx} \bigg|_{x=x_0} \) in Leibniz notation means the same thing as \( f'(x_0) \) in LaGrange notation. However, this still doesn’t tell us what \( dy/dx \) means. Leibniz’ intention was that \( dy/dx \) should mean the same thing as \( f' \). That is, \( dy/dx \) should be the derivative function of the function \( f \). Suffice it to say that these logical difficulties can be removed by slightly altering the Leibniz notation. The reader interested in the details of how this can be done should read the optional Section 3.

Having pointed out the pitfalls of the Leibniz notation, we will henceforth make the convention that if we have a function, \( y = f(x) \), then:

- \( dy/dx \) (Leibniz notation) means the same thing as \( f' \) (LaGrange notation).
- \( dy/dx \bigg|_{x=x_0} \) (Leibniz notation) means the same thing as \( f'(x_0) \) (LaGrange notation).

For example, the derivative of the function \( y = f(x) = x^2 \) is the function \( f'(x) = 2x \) (expressed in LaGrange notation). We express this same idea in Leibniz notation by writing

\[
\frac{dy}{dx} = 2x.
\]

If we want to express the fact that the derivative of \( f \) at \( x_0 = 3 \) is \( 2 \cdot 3 = 6 \), then this idea is expressed in LaGrange notation by writing

\[
f'(3) = 6
\]

and the same idea is expressed in Leibniz notation by writing

\[
\frac{dy}{dx} \bigg|_{x=3} = 6.
\]

As two more examples of the Leibniz notation, the fact that water depth, \( D = f(t) \), is increasing at the rate of 0.212 ft/min at time \( t = 5 \) min can be expressed by writing

\[
\frac{dD}{dt} \bigg|_{t=5} = 0.212,
\]

and the fact that the diameter, \( D = 2r \), of a circle increases at twice the rate that the radius increases (Example 6) is expressed by writing

\[
\frac{dD}{dr} = 2.
\]
Some more detailed examples that employ the Leibniz notation are given below.

**Example 7**  The area, $A$, of a circle of radius $r$ inches is

$$ A = \pi r^2 \text{ square inches}. $$

Hence, the area ($A$) is a function of the radius ($r$). The derivative of this function is

$$ \frac{dA}{dr} = 2\pi r. $$

Since the independent variable ($r$) has units of inches and the dependent variable ($A$) has units of square inches, the derivative has units of “square inches per inch” (or in$^2$ / in).

The fact that $\frac{dA}{dr} = 2\pi r$ means that if a circle has radius $r$ and we increase the radius by a very small amount, then the corresponding increase in area will be about $2\pi r$ times that amount. For example, the fact that

$$ \left. \frac{dA}{dr} \right|_{r=2} = 2\pi \cdot 2 = 4\pi \approx 12.566 \text{ in}^2 / \text{in} $$

means that if a circle has radius 2 in and we increase the radius by a very small amount -- say $h$ in, then the corresponding increase in area will be about

$$ (12.566 \text{ in}^2 / \text{in}) (h \text{ in}) = 12.566h \text{ in}^2. $$

**Example 8**  The Empire State Building in New York, which was constructed in 1931, is 1200 feet tall. If an object (let’s say a bowling ball) is dropped from the top of the Empire State Building, then the height ($h$) of the ball (in feet from ground level) at time $t$ seconds after the ball is dropped is

$$ h = -16t^2 + 1200. $$

(We will learn later how Calculus is used to arrive at the above formula for the height. This formula is valid only under the assumption that gravity is the only force acting on the falling ball. Other forces, such as air resistance are being ignored. Hence, this formula should be regarded only as an approximation to a realistic mathematical model for falling bodies.)

Since $t$ has units of “seconds” and $h$ has units of “feet”, the derivative,

$$ \frac{dh}{dt} = -32t, $$

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has units of “feet per second”. Hence, \( dh/dt \) tells us the velocity of the falling ball. For example, at time \( t = 1 \) sec, the ball has velocity

\[
\left. \frac{dh}{dt} \right|_{t=1} = -32 \cdot 1 = -32 \text{ ft/sec.}
\]

The fact that the velocity is negative means that the height of the ball is decreasing. In particular, the ball is falling at a speed of 32 ft/sec at time \( t = 1 \) sec. At time \( t = 2.5 \) sec, the velocity of the ball is

\[
\left. \frac{dh}{dt} \right|_{t=2.5} = -32 \cdot (2.5) = -80 \text{ ft/sec},
\]

which means that at time \( t = 2.5 \) sec, the ball is falling with speed 80 ft/sec. Note that the ball is falling faster and faster as it falls. This is because it is being accelerated by gravity.

### 2.1.1 Exercises

1. A cylindrical tank with radius 6 ft and height 18 ft is being filled with water. If \( V \) is the volume of water in the tank, and \( D \) is the depth of the water (in feet), then

\[
V = \pi (6)^2 (D) = 36\pi D.
\]

Hence, the volume of water in the tank is a function of the depth of the water,

\[
V = 36\pi D.
\]

(a) Compute \( dV/dD \).

(b) What are the units of \( D \), \( V \), and \( dV/dD \)?

(c) What does \( dV/dD \) tell us about the relationship between the volume and depth of water in the tank?

2. A cone–shaped tank with radius (at the top) 6 ft and height 18 ft is being filled with water. If \( V \) is the volume of water in the tank, \( r \) is the radius of the water surface (in feet) and \( D \) is the depth of the water (in feet), then

\[
V = \frac{1}{3} \pi r^2 D.
\]

(4)
Using the fact that
\[ \frac{r}{D} = \frac{6}{18}, \]
(because of similar triangles), we obtain
\[ r = \frac{1}{3} D. \]

Substitution into equation (4) then gives us
\[ V = \frac{1}{3} \pi \left( \frac{1}{3} D \right)^2 D = \frac{1}{27} \pi D^3. \]

Hence, the volume of water in the tank is a function of the depth of the water,
\[ V = \frac{1}{27} \pi D^3. \]

(a) Find \( dV/dD \).

(b) What are the units of \( D, V, \) and \( dV/dD? \)

(c) What does \( dV/dD \) tell us about the relationship between the volume and depth of water in the tank? In order to give a specific example, compute
\[ \left. \frac{dV}{dD} \right|_{D=8} \]
and explain what the value of this derivative tells us about the relationship between the water volume and water depth.

3. A cone–shaped tank with radius (at the top) 8 ft and height 18 ft is being filled with water at the rate of 10 gal/min (which is about 1.3368 ft\(^3\)/min). If \( D \) is the depth of the water (in feet), and \( t \) is time (in minutes), then
\[ D = kt^{1/3} \]
where \( k \) is a constant approximately equal to 1.8627. The derivative of \( D \) with respect to \( t \) is
\[ \frac{dD}{dt} = \frac{1}{3} kt^{-2/3}. \]

(a) What are the units of \( dD/dt? \)
(b) What does \( dD/dt \) tell us about the relationship between the depth of water in the tank and time? In order to give a specific example, compute

\[
\left. \frac{dD}{dt} \right|_{t=5}
\]

and explain what the value of this derivative tells us about the relationship between the water depth and time.

4. For each of the functions given in a–e, find the indicated derivative.

(a) \( y = -4x^3 + 6x^2 - 2x - 2 \). Find \( dy/dx \).

(b) \( s = 4t^6 - 5t^3 + 2t \). Find \( ds/dt \).

(c) \( y = 3x^3 - 2x + 12 \). Find \( dy/dx \) \( |_{x=3} \).

(d) \( z = -3t^3 - \frac{2}{3}t^2 + \frac{1}{5}t \). Find \( dz/dt \) \( |_{t=1} \).

(e) \( y = 9 \). Find \( dy/dx \).

5. If a bowling ball is thrown downward from the top of the Empire State Building with an initial speed of 80 feet per second, then the height \( (h) \) of the ball (in feet from ground level) at time \( t \) seconds after the ball is thrown is

\[
h = -16t^2 - 80t + 1200.
\]

(a) How fast is the ball travelling at time \( t = 1 \) sec?

(b) How fast is the ball travelling at time \( t = 2.5 \) sec?

(c) About how long will it take the ball to reach the ground and how fast is it going when it hits the ground?

3 \ A More Rigorous Development of the Leibniz Notation (Optional Reading)

When we write \( y = f(x) \), we are stating that \( f \) is a function whose independent variable is named \( x \) and whose dependent variable is named \( y \). If \( f \) is differentiable at a given point, \( x_0 \), then the derivative of \( f \) at \( x_0 \) (using the LaGrange notation) is the number \( f'(x_0) \) which we define as

\[
f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]  

(5)
The linear function, $L$, that best approximates $f$ near $x_0$ is

$$L(x) - f(x_0) = f'(x_0)(x - x_0) \quad (6)$$

and the graph of $L$ is called the tangent line to the graph of $f$ at the point $(x_0, f(x_0))$.

The formulation of the Leibniz notation for the derivative of $f$ at $x_0$ given in Section 2.1 is not entirely satisfactory because the symbols $\Delta x$, $\Delta y$, $dx$, and $dy$ used in that formulation do not make reference to the point, $x_0$, at which the derivative is being computed (even though the definitions of these symbols did make reference to $x_0$). A rigorous development of the Leibniz notation requires that we actually think of the quantities $\Delta x$, $\Delta y$, $dx$, and $dy$ as being functions of two variables, $x$ and $x_0$. The quantities $\Delta x$ and $\Delta y$ are called differences and the quantities $dx$ and $dy$ are called differentials. The main change we need to make is to change the notations for the differences and differentials to $\Delta x(x_0, x)$, $\Delta y(x_0, x)$, $dx(x_0, x)$, and $dy(x_0, x)$ to emphasize the fact that these differences and differentials also depend on two points, $x_0$ and $x$.

Given a function, $y = f(x)$, that is differentiable at the given point $x_0$, and with $f'(x_0)$ and $L$ as defined in equations (5) and (6), we define

- $\Delta x(x_0, x) = x - x_0$ where $x$ is any point in the domain of $f$ with $x \neq x_0$.
- $\Delta y(x_0, x) = f(x) - f(x_0)$.
- $dx(x_0, x) = x - x_0$. (Note that $dx(x_0, x) = \Delta x(x_0, x)$.)
- $dy(x_0, x) = L(x) - L(x_0)$. (Note that since $L(x_0) = f(x_0)$, we can also write $dy(x_0, x) = L(x) - f(x_0)$.)

Having given these more formal definitions of differences and differentials, we now note that

$$f'(x_0) = \lim_{x \to x_0} \frac{\Delta y(x_0, x)}{\Delta x(x_0, x)} \quad \text{(by definition of the derivative)}$$

and

$$f'(x_0) = \frac{L(x) - L(x_0)}{x - x_0} = \frac{dy(x_0, x)}{dx(x_0, x)}$$

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by virtue of the fact that the slope of the tangent line can be computed by using the point \((x_0, L(x_0))\) and any other point \((x, L(x))\) on the tangent line.

The result we obtain is thus

\[
\frac{dy(x_0, x)}{dx(x_0, x)} = \lim_{x \to x_0} \frac{\Delta y(x_0, x)}{\Delta x(x_0, x)}.
\] (7)

Equation (7) is formally correct because it actually allows us to interpret \(dy(x_0, x)\) and \(dx(x_0, x)\) as numbers and to interpret \(dy(x_0, x)/dx(x_0, x)\) as a bona-fide ratio. Since the value of this ratio does not depend on the choice of \(x \neq x_0\), we can define \(dy/dx\) to be the function which assigns each value \(x_0\) (at which \(f\) is differentiable) to the value \(dy(x_0, x)/dx(x_0, x)\) (which, recall, does not depend on \(x\)). In other words, we can define \(dy/dx\) to be the function defined by

\[
\frac{dy}{dx}(x_0) = \lim_{x \to x_0} \frac{\Delta y(x_0, x)}{\Delta x(x_0, x)}
\]

for all points \(x_0\) at which \(f\) is differentiable. It is equally common to write \(\frac{dy}{dx}\big|_{x=x_0}\) instead of \(\frac{dy}{dx}(x_0)\). The main thing to understand here is that \(dy\) and \(dx\) are each functions of two variables \((x_0\) and \(x\), meaning that \(dy/dx\) (in the usual sense of the ratio of two functions) is also a function of two variables. However, since the value of \(dy/dx\) depends only on \(x_0\), we can regard \(dy/dx\) (now not thought of as a ratio in the usual sense) as a function of the single variable \(x_0\).