Homework Assignment 16–Solutions

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1. Find the magnitude and the direction angles of the vector \( \overrightarrow{v} = (1, -6, 2) \).

**Solution:** The magnitude of \( \overrightarrow{v} \) is

\[
|\overrightarrow{v}| = \sqrt{1^2 + (-6)^2 + 2^2} = \sqrt{41}
\]

The direction angles of \( \overrightarrow{v} \) are \( \alpha, \beta, \) and \( \gamma \) (all between 0 and \( \pi \)) such that

\[
\cos \alpha = \frac{1}{\sqrt{41}} \\
\cos \beta = \frac{-6}{\sqrt{41}} \\
\cos \gamma = \frac{2}{\sqrt{41}}
\]

Taking inverse cosines, we obtain

\[
\alpha = \arccos \left( \frac{1}{\sqrt{41}} \right) \approx 1.41398 \, 0641
\]

\[
\beta = \arccos \left( \frac{-6}{\sqrt{41}} \right) \approx 2.78485 \, 9265
\]

\[
\gamma = \arccos \left( \frac{2}{\sqrt{41}} \right) \approx 1.25313 \, 3134
\]

Converting from radians to degrees, we obtain

\[
\alpha \approx 81.01512 \, 303^\circ \\
\beta \approx 159.56068 \, 24^\circ \\
\gamma \approx 71.79923 \, 972^\circ
\]
2. Find the vector, \( \overrightarrow{v} \), which has magnitude 4 and direction angles \( \alpha = \pi/2, \beta = \pi/2, \) and \( \gamma = 0 \).

Solution:

\[
\overrightarrow{v} = |\overrightarrow{v}| \left( \cos \alpha \overrightarrow{i} + \cos \beta \overrightarrow{j} + \cos \gamma \overrightarrow{k} \right) \\
= 4 \left( \cos \left( \frac{\pi}{2} \right) \overrightarrow{i} + \cos \left( \frac{\pi}{2} \right) \overrightarrow{j} + \cos (0) \overrightarrow{k} \right) \\
= 4 \overrightarrow{k}
\]

3. Find three different representatives of the vector \( \overrightarrow{v} = (4, -2, 0) \). (Recall that a representative of \( \overrightarrow{v} \) is a directed line segment with same magnitude and direction as \( \overrightarrow{v} \). See Section 1.1 of my notes.) What is the standard representative of \( \overrightarrow{v} \)?

Solution: We take the following points

\[
P_0 (0, 0, 0), \quad P_1 (4, -2, 0) \\
Q_0 (3, 3, 2), \quad Q_1 (7, 1, 2) \\
R_0 (-4, 0, \sqrt{5}), \quad R_1 (0, -2, \sqrt{5})
\]

The directed line segments from \( P_0 \) to \( P_1 \), from \( Q_0 \) to \( Q_1 \), and from \( R_0 \) to \( R_1 \) are all representatives of \( \overrightarrow{v} \). The standard representative of \( \overrightarrow{v} \) is the directed line segment from \( P_0 \) to \( P_1 \).

4. Let \( \overrightarrow{v} = (9, -3, -6) \). Find the representative of \( \overrightarrow{v} \) which has initial point at \( P_0 (0, -12, 4) \). Find the representative of \( \overrightarrow{v} \) which has terminal point at \( P_1 (4, 0, -10) \). Find the standard representative of \( \overrightarrow{v} \).

Solution: The representative of \( \overrightarrow{v} \) which has initial point at \( P_0 (0, -12, 4) \) has terminal point at \( P_1 (9, -15, -2) \). The representative of \( \overrightarrow{v} \) which has terminal point at \( P_1 (4, 0, -10) \) has initial point at \( P_0 (-5, 3, -4) \).

5. Let \( \overrightarrow{u} = (2, 6, 0) \), \( \overrightarrow{v} = (1, 1, 4) \), and \( t = -3 \). Compute \( \overrightarrow{u} + \overrightarrow{v} \) and \( t \overrightarrow{v} \).

Solution:

\[
\overrightarrow{u} + \overrightarrow{v} = 3 \overrightarrow{i} + 7 \overrightarrow{j} + 4 \overrightarrow{k} \\
t \overrightarrow{v} = -3 \overrightarrow{i} - 3 \overrightarrow{j} - 12 \overrightarrow{k}
\]
6. Let $\vec{u} = (5, 3/2, 4/5)$ and $\vec{v} = (6, -6, 1/5)$. Compute $2\vec{u} - 5\vec{v}$.

**Solution:**

$$2\vec{u} - 5\vec{v} = \left(10\vec{i} + 3\vec{j} + \frac{8}{5}\vec{k}\right) - \left(30\vec{i} - 30\vec{j} + \vec{k}\right)$$

$$= -20\vec{i} + 33\vec{j} + \frac{3}{5}\vec{k}$$

7. Let $\vec{v} = (9, 2, -1/2)$. Find a vector whose magnitude is four times the magnitude of $\vec{v}$ and which points in the opposite direction of $\vec{v}$. Also, find a vector whose magnitude is one tenth the magnitude of $\vec{v}$ and which points in the same direction as $\vec{v}$.

**Solution:** A vector whose magnitude is four times the magnitude of $\vec{v}$ and which points in the opposite direction of $\vec{v}$ is

$$-4\vec{v} = -36\vec{i} - 8\vec{j} + 2\vec{k}$$

A vector whose magnitude is one tenth the magnitude of $\vec{v}$ and which points in the same direction as $\vec{v}$ is

$$\frac{1}{10}\vec{v} = \frac{9}{10}\vec{i} + \frac{1}{5}\vec{j} - \frac{1}{20}\vec{k}$$

8. Is $\vec{u} = (1/4, 1/2, -1/4)$ a unit vector?

**Solution:** We have

$$|\vec{u}| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2}$$

$$= \sqrt{\frac{1}{16} + \frac{1}{4} + \frac{1}{16}}$$

$$= \sqrt{\frac{3}{8}}$$

Since $|\vec{u}| \neq 1$, then, by definition, $\vec{u}$ is not a unit vector.

9. Find a unit vector which points in the direction of $\vec{v} = (9, 2, -1/2)$. 
Solution: We have
\[
|\mathbf{v'}| = \sqrt{9^2 + 2^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{85 + \frac{1}{4}} = \sqrt{\frac{341}{4}} = \frac{1}{2} \sqrt{341}
\]

A unit vector in the direction of \(\mathbf{v'}\) is
\[
\frac{1}{|\mathbf{v'}|} \mathbf{v'} = \frac{2}{\sqrt{341}} \left(9 \mathbf{i} + 2 \mathbf{j} - \frac{1}{2} \mathbf{k}\right) = \frac{18}{\sqrt{341}} \mathbf{i} + \frac{4}{\sqrt{341}} \mathbf{j} - \frac{1}{\sqrt{341}} \mathbf{k}
\]

10. Given that \(\mathbf{u'}\) and \(\mathbf{v'}\) are vectors with \(|\mathbf{u'}| = 2\), \(\mathbf{u'} \cdot \mathbf{v'} = 3\), and the angle between \(\mathbf{u'}\) and \(\mathbf{v'}\) is \(\theta = \pi/4\), find \(|\mathbf{v'}|\).
Solution: Since
\[
\cos \theta = \frac{\mathbf{u'} \cdot \mathbf{v'}}{|\mathbf{u'}| |\mathbf{v'}|},
\]
we have
\[
|\mathbf{v'}| = \frac{\mathbf{u'} \cdot \mathbf{v'}}{|\mathbf{u'}| \cos \theta} = \frac{3\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}.
\]

11. Use the dot product to show that the vectors \(\mathbf{u'} = 5 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}\) and \(\mathbf{v'} = 2 \mathbf{i} - 3 \mathbf{j} - 16 \mathbf{k}\) are perpendicular.
Solution:
\[
\mathbf{u'} \cdot \mathbf{v'} = (5)(2) + (-2)(-3) + (1)(-16) = 0
\]
so \(\mathbf{u'}\) and \(\mathbf{v'}\) are perpendicular.

12. Let \(\mathbf{u'} = 4 \mathbf{i} + \frac{1}{2} \mathbf{j}\) and \(\mathbf{v'} = 2 \mathbf{i} - \frac{1}{2} \mathbf{k}\). Find the sine of the angle, \(\theta\), between \(\mathbf{u'}\) and \(\mathbf{v'}\). (Hint: You should easily be able to obtain \(\cos \theta\) by using the dot product.) Also find \(|\mathbf{u'}|\), \(|\mathbf{v'}|\), and \(|\mathbf{u'} \times \mathbf{v'}|\). Then,
verify that \(|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin \theta\). Do all of this without using a 
calculator or computer.

**Solution:** We have
\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{8}{\sqrt{65} \cdot \sqrt{17}} = \frac{32}{\sqrt{1105}}
\]
which gives us
\[
\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left( \frac{32}{\sqrt{1105}} \right)^2 = 1 - \frac{32^2}{1105 - 1024} = \frac{81}{1105}
\]
Since \(0 < \theta < \pi\), we know that \(\sin \theta > 0\). Thus,
\[
\sin \theta = \sqrt{\frac{81}{1105}}
\]
Now, we have
\[
|\vec{u}| = \frac{1}{2} \sqrt{65}
\]
\[
|\vec{v}| = \frac{1}{2} \sqrt{17}
\]
and
\[
\vec{u} \times \vec{v} = \left(4 \vec{i} + \frac{1}{2} \vec{j}\right) \times \left(2 \vec{i} - \frac{1}{2} \vec{k}\right)
\]
\[
= 4 \vec{i} \times \left(2 \vec{i} - \frac{1}{2} \vec{k}\right) + \frac{1}{2} \vec{j} \times \left(2 \vec{i} - \frac{1}{2} \vec{k}\right)
\]
\[
= 4 \vec{i} \times 2 \vec{i} - 4 \vec{i} \times \frac{1}{2} \vec{k} + \frac{1}{2} \vec{j} \times 2 \vec{i} - \frac{1}{2} \vec{j} \times \frac{1}{2} \vec{k}
\]
\[
= 0 - 2(- \vec{j}) + (- \vec{k}) - \frac{1}{4} \vec{i}
\]
\[
= -\frac{1}{4} \vec{i} + 2 \vec{j} - \vec{k}
\]
which gives
\[
|\vec{u} \times \vec{v}| = \sqrt{\left(-\frac{1}{4}\right)^2 + 2^2 + (-1)^2}
\]
\[
= \sqrt{\frac{81}{16}}
\]
\[
= \frac{1}{4}\sqrt{81}.
\]
Now we can verify that the equation
\[
|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta
\]
is correct. We have
\[
|\vec{u}| |\vec{v}| \sin \theta = \frac{1}{2} \sqrt{65} \cdot 1 \cdot \sqrt{\frac{81}{1105}}
\]
\[
= \frac{1}{4} \sqrt{65 \cdot 17 \cdot 81}
\]
\[
= \frac{1}{4} \sqrt{81}
\]

13. Use the cross product to verify that the vectors \(\vec{u} = 7\vec{i} - 3\vec{j} - \frac{4}{3}\vec{k}\) and \(\vec{v} = -21\vec{i} + 9\vec{j} + 4\vec{k}\) are parallel. Do you see an easier way to see that \(\vec{u}\) and \(\vec{v}\) are parallel?

**Solution:** Since \(\vec{u} \times \vec{v} = 0\), we know that \(\vec{u}\) and \(\vec{v}\) are parallel. A quicker way to see this (without going through all of the work to compute the cross product) is just to observe that \(\vec{v} = -3\vec{u}\).

14. Find parametric equations for the line, \(L_1\), which contains the points \(P(0, -6, 4)\) and \(Q(1, 1, -6)\). Once you have done this, verify that the point \(P_1(-3, -27, 34)\) is on \(L_1\) but the point \(P_2(-1, -10, 14)\) is not on \(L_1\).

**Solution:** A direction vector for \(L_1\) is
\[
\vec{v}_1 = \overrightarrow{PQ} = \vec{i} + 7\vec{j} - 10\vec{k}.
\]
Parametric equations for $L_1$ are

\[
\begin{align*}
x &= t \\
y &= -6 + 7t \\
z &= 4 - 10t 
\end{align*}
\]

If we set $t = -3$, we obtain

\[
\begin{align*}
x &= -3 \\
y &= -27 \\
z &= 34 
\end{align*}
\]

which shows that $P_1$ is on $L_1$. In order to have $x = -1$, we must have $t = -1$ but this would give us that $y = -13$. This shows that $P_2$ is not on $L_1$.

15. Find parametric equations for the line, $L_2$, which is parallel to the line, $L_1$, of the previous problem, and which contains the point $P_2 (-1, -10, 14)$.

**Solution:** Parametric equations for $L_2$ are

\[
\begin{align*}
x &= -1 + t \\
y &= -10 + 7t \\
z &= 14 - 10t 
\end{align*}
\]

16. Find an equation for the plane, $\Pi_1$, which contains the points $P (0, 0, -7)$, $Q (1, 2, -4)$, and $R (-1, 0, 1)$. After you have done this, verify that the point $P_1 (2, 1, -35/2)$ is on $\Pi_1$ but that the point $P_2 (5, -3/2, 3/2)$ is not on $\Pi_1$.

**Solution:** A normal vector for $\Pi_1$ is

\[
\begin{align*}
\vec{n} &= \vec{PQ} \times \vec{PR} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 0 & 8 \end{vmatrix} \\
&= 16 \vec{i} - 11 \vec{j} + 2 \vec{k}.
\end{align*}
\]

An equation for $\Pi_1$ is

\[
16(x - 0) - 11(y - 0) + 2(z - (-7)) = 0
\]
or

\[ 16x - 11y + 2z = -14. \]

Since

\[ 16 (2) - 11 (1) + 2 (-35/2) = -14, \]

we see that \( P_1 \) is on \( \Pi_1 \). Since

\[ 16 (5) - 11 (-3/2) + 2 (3/2) \neq -14, \]

we see that \( P_2 \) is on \( \Pi_1 \).

17. Find an equation for a plane, \( \Pi_2 \), which is parallel to the plane, \( \Pi_1 \), of the previous problem and which contains the point \( P_2 \) \((5, -3/2, 3/2)\).

**Solution:** An equation for \( \Pi_2 \) is

\[ 16 (x - 5) - 11 \left( y + \frac{3}{2} \right) + 2 \left( z - \frac{3}{2} \right) = 0. \]

(Note that parallel planes have parallel normal vectors.)

18. Let \( L \) be the line with parametric equations

\[
\begin{align*}
  x &= 6 - t \\
  y &= 4 - 2t \\
  z &= 3
\end{align*}
\]

and let \( P \) be the point \( P (0, 0, 4) \). Find the distance from \( P \) to \( L \).

**Solution:** The line \( L \) contains the point \( P_0 \) \((6, 4, 3)\) and has direction vector \( \vec{v}' = -i - 2j \). The distance from \( P \) to \( L \) is

\[
\frac{|P_0 \vec{P} \times \vec{v}'|}{|\vec{v}'|}.
\]

We have

\[
\vec{P_0 \vec{P}} \times \vec{v}' = -16i + 24j
\]

from which we obtain

\[
|P_0 \vec{P} \times \vec{v}'| = \sqrt{16^2 + 24^2} = 8\sqrt{13}.
\]

Also, \( |\vec{v}'| = \sqrt{5} \) so the distance from \( P \) to \( L \) is \( 8\sqrt{13}/\sqrt{5} \).
19. Find the distance between the point \( P(0, 0, 4) \) and the plane \( \Pi : 3x + 2z = 0 \).

**Solution:** The plane \( \Pi \) has normal vector \( \vec{n}' = 3\vec{i} + 2\vec{k} \). A point on \( \Pi \) is \( P_0(0, 0, 0) \). This distance between \( P \) and \( \Pi \) is

\[
\frac{|\vec{P_0P} \cdot \vec{n}'|}{|\vec{n}'|} = \frac{8}{\sqrt{13}}
\]

20. Let \( L_1 \) be the line with parametric equations

\[
\begin{align*}
x &= 6 - t \\
y &= 4 - 2t \\
z &= 2 + t
\end{align*}
\]

and let \( L_2 \) be the line with parametric equations

\[
\begin{align*}
x &= 5 - 5t \\
y &= 2 - 10t \\
z &= 3 + 5t
\end{align*}
\]

Show that \( L_1 \) and \( L_2 \) are the same line.

**Solution:** \( L_1 \) has direction vector \( \vec{v}_1 = -\vec{i} - 2\vec{j} + \vec{k} \) and \( L_2 \) has direction vector \( \vec{v}_2 = -5\vec{i} - 10\vec{j} + 5\vec{k} \). Since \( \vec{v}_1 \times \vec{v}_2 = \vec{0} \), we see that \( L_1 \) and \( L_2 \) are parallel. Also, \( L_1 \) contains the point \( P_1(6, 4, 2) \) and \( L_2 \) contains the point \( P_2(5, 2, 3) \). Since \( P_1P_2 \times \vec{v}_1 = \vec{0} \), we see that \( L_1 \) and \( L_2 \) are, in fact, the same line.

21. Let \( L_1 \) be the line with parametric equations

\[
\begin{align*}
x &= 6 - t \\
y &= 4 - 2t \\
z &= 2 + t
\end{align*}
\]

and let \( L_2 \) be the line with parametric equations

\[
\begin{align*}
x &= 4 + 3t \\
y &= 4 + 6t \\
z &= 4 - 3t
\end{align*}
\]
Show that $L_1$ and $L_2$ are parallel (but not the same line) and find the distance between $L_1$ and $L_2$.

**Solution:** $L_1$ has direction vector $\vec{v}_1 = -\vec{i} - 2\vec{j} + \vec{k}$ and $L_2$ has direction vector $\vec{v}_2 = 3\vec{i} + 6\vec{j} - 3\vec{k}$. Since $\vec{v}_1 \times \vec{v}_2 = \vec{0}$, we see that $L_1$ and $L_2$ are parallel. Also, $L_1$ contains the point $P_1(6,4,2)$ and $L_2$ contains the point $P_2(4,4,4)$. We now compute

$$\vec{P}_1\vec{P}_2 \times \vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ -1 & -2 & 1 \end{vmatrix} = 4\vec{i} + 4\vec{k}$$

Since $\vec{P}_1\vec{P}_2 \times \vec{v}_1 \neq \vec{0}$, we see that $L_1$ and $L_2$ are parallel but not the same. Also, the distance between $L_1$ and $L_2$ is

$$\frac{|\vec{P}_1\vec{P}_2 \times \vec{v}_1|}{|\vec{v}_1|} = \frac{4\sqrt{2}}{\sqrt{6}} = \frac{4\sqrt{3}}{3}$$

22. Let $L_1$ be the line with parametric equations

$$x = 6 - t$$
$$y = 4 - 2t$$
$$z = 2 + t$$

and let $L_2$ be the line with parametric equations

$$x = 1 + 2t$$
$$y = 4 - 6t$$
$$z = 4 + t$$

Show that $L_1$ and $L_2$ intersect at a single point and find this point.

**Solution:** $L_1$ has direction vector $\vec{v}_1 = -\vec{i} - 2\vec{j} + \vec{k}$ and $L_2$ has direction vector $\vec{v}_2 = 2\vec{i} - 6\vec{j} + \vec{k}$. We have

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ 2 & -6 & 1 \end{vmatrix} = 4\vec{i} + 3\vec{j} + 10\vec{k}$$
Since \( \vec{v}_1 \times \vec{v}_2 \neq \vec{0} \), we see that \( L_1 \) and \( L_2 \) are not parallel. Also, \( L_1 \) contains the point \( P_1 (6, 4, 2) \) and \( L_2 \) contains the point \( P_2 (1, 4, 4) \) and we have
\[
\vec{P}_1 P_2 \cdot (\vec{v}_1 \times \vec{v}_2) = \left( -5 \vec{i} + 2 \vec{k} \right) \cdot \left( 4 \vec{i} + 3 \vec{j} + 10 \vec{k} \right) = 0.
\]
This shows that \( L_1 \) and \( L_2 \) intersect at a single point. To find the point of intersection, we need to find real numbers \( t \) and \( s \) such that
\[
\begin{align*}
6 - t &= 1 + 2s \\
4 - 2t &= 4 - 6s \\
2 + t &= 4 + s
\end{align*}
\]
Rearrangement of these equations gives
\[
\begin{align*}
t + 2s &= 5 \\
2t - 6s &= 0 \\
t - s &= 2
\end{align*}
\]
The only solution to this system of equations is \( t = 3 \), \( s = 1 \). Hence, the lines \( L_1 \) and \( L_2 \) intersect at the point \((3, -2, 5)\).

23. Let \( L_1 \) be the line with parametric equations
\[
\begin{align*}
x &= 6 - t \\
y &= 4 - 2t \\
z &= 2 + t
\end{align*}
\]
and let \( L_2 \) be the line with parametric equations
\[
\begin{align*}
x &= 4 + 2t \\
y &= -6t \\
z &= 4
\end{align*}
\]
Show that \( L_1 \) and \( L_2 \) are skew and find the distance between \( L_1 \) and \( L_2 \).

**Solution:** \( L_1 \) has direction vector \( \vec{v}_1 = -\vec{i} - 2\vec{j} + \vec{k} \) and \( L_2 \) has direction vector \( \vec{v}_2 = 2\vec{i} - 6\vec{j} \). We have
\[
\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ 2 & -6 & 0 \end{vmatrix} = 6\vec{i} + 2\vec{j} + 10\vec{k}
\]
Since $\vec{v}_1 \times \vec{v}_2 \neq \vec{0}$, we see that $L_1$ and $L_2$ are not parallel. Also, $L_1$ contains the point $P_1(6, 4, 2)$ and $L_2$ contains the point $P_2(4, 0, 4)$ and we have

$$\overrightarrow{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) = \left( -2 \vec{i} - 4 \vec{j} + 2 \vec{k} \right) \cdot \left( 6 \vec{i} + 2 \vec{j} + 10 \vec{k} \right) = -12 - 8 + 20 = 0.$$

Hence, $L_1$ and $L_2$ intersect at a single point. (Whoops, they are not skew as stated in the statement of the problem!) Let us find the point where $L_1$ and $L_2$ intersect. We need to find real numbers $t$ and $s$ such that

$$6 - t = 4 + 2s$$
$$4 - 2t = -6s$$
$$2 + t = 4$$

The only solution to this system of equations is $t = 2$, $s = 0$. The lines $L_1$ and $L_2$ intersect at the point $(4, 0, 4)$.

24. Find the area of the triangle with vertices at the points $P(0, -4, 0)$, $Q(1, 2, -10)$, and $R(0, 2, 7)$.

**Solution:** The area of this triangle is

$$\frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|.$$ 

First, we compute the cross product:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 6 & -10 \\
0 & 6 & 7 \\
\end{vmatrix} = 102 \vec{i} - 7 \vec{j} + 6 \vec{k}.$$ 

This gives

$$\frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \sqrt{102^2 + (-7)^2 + 6^2} = \sqrt{10,489}.$$ 

The area of the given triangle is $\frac{1}{2} \sqrt{10,489} \approx 102.4158191$. 

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