Solutions for Homework Assignment 27

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1. According to our results, the right circular cylinder which has surface area 10 square feet and which has largest possible volume is one with

\[ \text{radius} = \sqrt{\frac{10}{6\pi}} \approx 0.7283656203 \text{ ft} \]

and

\[ \text{height} = 2\sqrt{\frac{10}{6\pi}} \approx 1.456731241 \text{ ft.} \]

The volume of this cylinder is

\[ \pi r^2 h = \pi \left( \sqrt{\frac{10}{6\pi}} \right)^2 \left( 2\sqrt{\frac{10}{6\pi}} \right) = 2\pi \left( \frac{10}{6\pi} \right)^{3/2} \approx 2.427885403 \text{ cubic feet.} \]

2. Assuming that our cylinder has no top or bottom, the function to be maximized is

\[ V(r, h) = \pi r^2 h \]

and the constraint condition is

\[ g(r, h) = 2\pi rh = A. \]

The LaGrange system to solve is thus

\[
\begin{align*}
2\pi rh &= \lambda (2\pi h) \\
\pi r^2 &= \lambda (2\pi r) \\
2\pi rh &= A
\end{align*}
\]
which simplifies to

\[ r = \lambda \]
\[ r = 2\lambda \]
\[ 2\pi rh = A \]

In order for both of the first two equations to be satisfied, we must have \( \lambda = 0 \), but this implies that \( r = 0 \), and the last equation then implies that \( A = 0 \). However, we are assuming that \( A > 0 \) so the system has no solution. We conclude that there is no cylinder of maximum volume for a fixed surface area \( A \) (assuming that the cylinder has no top or bottom).

3. Assuming that our cylinder has a bottom but no top, the function to be maximized is

\[ V(r, h) = \pi r^2 h \]

and the constraint condition is

\[ g(r, h) = 2\pi rh + \pi r^2 = A. \]

The LaGrange system to solve is thus

\[ 2\pi rh = \lambda (2\pi h + 2\pi r) \]
\[ \pi r^2 = \lambda (2\pi r) \]
\[ 2\pi rh + \pi r^2 = A \]

which simplifies to

\[ rh = \lambda (h + r) \]
\[ r = 2\lambda \]
\[ 2\pi rh + \pi r^2 = A \]

The second equation in this system tells us that \( \lambda = \frac{1}{2}r \). Substituting this in the first equation gives

\[ rh = \frac{1}{2}r (h + r) \]

or

\[ \frac{1}{2} rh = \frac{1}{2}r^2 \]

\[ 2 \]
or \( r = h \). Substitution of \( r = h \) into the last equation of the LaGrange system gives

\[
2\pi r^2 + \pi r^2 = A
\]

or

\[
3\pi r^2 = A.
\]

We thus obtain the solution

\[
\begin{align*}
r &= \sqrt{\frac{A}{3\pi}} \\
h &= \sqrt{\frac{A}{3\pi}}
\end{align*}
\]

which are the radius and height of the cylinder of maximum volume.

4. We want to verify that the numbers

\[
\begin{align*}
x &= \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \\
y &= \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \\
z &= \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \\
\lambda &= \frac{\sqrt{15}}{60p} \sqrt{\frac{D}{p}}
\end{align*}
\]

satisfy the system of equations

\[
\begin{align*}
yz &= \lambda (5py + 2pz) \\
xz &= \lambda (5px + 2pz) \\
xy &= \lambda (2px + 2py) \\
5pxy + 2pxz + 2pyz &= D
\end{align*}
\]

Let us check the equations one at a time:

\[
yz = \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) = \frac{D}{6p}
\]
and
\[
\lambda (5py + 2pz) = \frac{\sqrt{15}}{60p} \sqrt{\frac{D}{p}} \left( 5p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) + 2p \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) \right) = \frac{D}{6p}
\]
so equation (1) is satisfied.

\[
xz = \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) = \frac{D}{6p}
\]
and
\[
\lambda (5px + 2pz) = \frac{\sqrt{15}}{60p} \sqrt{\frac{D}{p}} \left( 5p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) + 2p \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) \right) = \frac{D}{6p}
\]
so equation (2) is satisfied.

\[
xy = \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) = \frac{D}{15p}
\]
and
\[
\lambda (2px + 2py) = \frac{\sqrt{15}}{60p} \sqrt{\frac{D}{p}} \left( 2p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) + 2p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \right) = \frac{D}{15p}
\]
so equation (3) is satisfied.

\[
5p \times y + 2pxz + 2pyz = 5p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) + 2p \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{15} \sqrt{\frac{D}{p}} \right) + 2p \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) \left( \frac{\sqrt{15}}{6} \sqrt{\frac{D}{p}} \right) = \frac{D}{6p}
\]
so equation (4) is satisfied.

5. Assuming that \( p = 10 \) dollars per square foot and that our budget is \( D = 400 \) dollars, the largest aquarium we can build is one with dimensions

\[
x = \frac{\sqrt{15}}{15} \sqrt{\frac{400}{10}} \approx 1.632993161 \text{ ft}
\]
\[
y = \frac{\sqrt{15}}{15} \sqrt{40} \approx 1.632993161 \text{ ft}
\]
\[
z = \frac{\sqrt{15}}{6} \sqrt{40} \approx 4.082482903 \text{ ft}
\]
The volume of this aquarium will be

\[ xyz = \left( \frac{\sqrt{15}}{15} \sqrt{40} \right) \left( \frac{\sqrt{15}}{15} \sqrt{40} \right) \left( \frac{\sqrt{15}}{6} \sqrt{40} \right) \approx 10.88662 \, 108 \text{ cubic feet.} \]

We want to convert cubic feet to gallons. There are several sites on the Web which allow you to convert units interactively - so if you are connected to the internet, you might want to check out one of these sites such as http://www.graniterock.com/calconv.htm. The conversion factor is

1 cubic foot = 7.4827 gallons.

Hence, our aquarium will have volume

\[ (10.88662 \, 108) (7.4827) \approx 81.46131 \, 95 \text{ gallons.} \]

It appears that we can accommodate about 27 goldfish but I hope that they like to swim vertically because the aquarium is about 1.6 foot square at the bottom by about 4 foot tall.

6. Assuming that slate and glass both have a price of \( p \) dollars per square foot, we want to maximize the function

\[ V(x, y, z) = xyz \]

subject to the constraint

\[ g(x, y, z) = px + 2pxz + 2pwy = D. \]

Using LaGrange multipliers, we must solve

\[ yz = \lambda (py + 2pz) \]
\[ xz = \lambda (px + 2pz) \]
\[ xy = \lambda (2px + 2py) \]
\[ px + 2pxz + 2pwy = D \]

Using Maple, we obtain the solution

\[ x = \frac{\sqrt{3}}{3} \sqrt[3]{\frac{D}{p}} \]
\[
y = \frac{\sqrt{3}}{3} \sqrt[3]{\frac{D}{p}}
\]
\[
z = \frac{\sqrt{3}}{6} \sqrt[3]{\frac{D}{p}}
\]
\[
\lambda = \frac{\sqrt{3}}{12p} \sqrt[3]{\frac{D}{p}}
\]

which tells us the dimensions of the largest (volume) aquarium we can build given our cost constraints. Assuming that slate (and glass) cost 10 dollars per square foot and assuming that we have 400 dollars to spend, we will construct an aquarium of volume

\[
xyz = \left( \frac{\sqrt{3}}{3} \sqrt[3]{\frac{D}{p}} \right) \left( \frac{\sqrt{3}}{3} \sqrt[3]{\frac{D}{p}} \right) \left( \frac{\sqrt{3}}{6} \sqrt[3]{\frac{D}{p}} \right) = \frac{\sqrt{3}}{18} \left( \frac{D}{p} \right)^{3/2} = \frac{\sqrt{3}}{18} (40)^{3/2} \approx 24.34322478 \text{ ft}^3.
\]

This is the same as

\[
(24.34322478) (7.4827) = 182.1530481 \text{ gallons.}
\]

We can accommodate about 60 goldfish.

7. This has already been done in the solutions to Homework Assignment 25.

8. We want to maximize

\[
f(x, y) = x^2 - 2xy + 2y
\]

subject to the constraint

\[
g(x, y) = y - x = 0.
\]

The LaGrange system to solve is

\[
2x - 2y = -\lambda
\]
\[
-2x + 2 = \lambda
\]
\[
y - x = 0
\]

The solution to this system is \(\lambda = 0, x = 1, y = 1\). Hence \(f(1, 1) = 1\) is the maximum value of \(f\) along the line \(y = x\).
An alternate way to do this problem is just to note that along the line 
\( y = x \), we have

\[
f(x, y) = -x^2 + 2x, \ 0 \leq x \leq 2
\]

and this function of one variable \( x \) has an absolute maximum value 
of 1 at \( x = 1 \).