Answers and Solutions to Homework Problems from Section 10.1 of Grossman
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1.) The first five terms of the sequence \(1/3^n\) are
\[
\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}
\]

3.) The first five terms of the sequence \(1 - 1/4^n\) are
\[
\frac{3}{4}, \frac{15}{16}, \frac{63}{64}, \frac{255}{256}, \frac{1023}{1024}
\]

7.) \(n/(n + 1)\)

10.) \(n/(2n + 1)\)

11.)
\[
\frac{17}{\sqrt{n}} \to 0.
\]

14.) \(\sin(n\pi) = 0\) for all \(n = 1, 2, 3, \ldots\), so, clearly, \(\sin(n\pi) \to 0\).

16.)
\[
\frac{n^3}{n^3 + 1} = \frac{1}{1 + \frac{1}{n^3}}
\]
for all sufficiently large \(n\), so
\[
\frac{n^3}{n^3 + 1} \to 1.
\]

Thus, if \(n\) is large and even,
\[
\frac{(-1)^n n^3}{n^3 + 1}
\]

is close to 1, whereas if \(n\) is large and odd,
\[
\frac{(-1)^n n^3}{n^3 + 1}
\]

is close to \(-1\). We conclude that
\[
\frac{(-1)^n n^3}{n^3 + 1}
\]
diverges (by oscillation).

17.)
\[
\lim_{n \to \infty} \frac{n^5 + 3n^2 + 1}{n^6 + 4n} = \lim_{n \to \infty} \frac{1 + \frac{3}{n^3} + \frac{1}{n^6}}{1 + \frac{4}{n^5}} = 0.
\]

22.) For \(n\) sufficiently large,
\[
\sqrt{n + 3} - \sqrt{n} = \left(\sqrt{n + 3} - \sqrt{n}\right) \cdot \frac{\sqrt{n + 3} + \sqrt{n}}{\sqrt{n + 3} + \sqrt{n}}
\]
\[
= \frac{3}{\sqrt{n + 3} + \sqrt{n}}
\]
so
\[ \lim_{n \to \infty} \left( \sqrt{n + 3} - \sqrt{n} \right) = \lim_{n \to \infty} \frac{3}{\sqrt{n + 3} + \sqrt{n}} = 0. \]

23.) First note that
\[ \frac{2^n}{n!} = \frac{2^n}{n!} \cdot \frac{2!}{2!} \]
Next, note that if \( n = 2 + k \) where \( k \) is a positive integer, then
\[ \frac{2!}{n!} = \frac{1 \cdot 2}{1 \cdot 2 \cdot (2 + 1) \cdot (2 + 2) \cdot \cdots \cdot (2 + k)} \]
\[ \leq \frac{1}{(2 + 1) \cdot (2 + 2) \cdot \cdots \cdot (2 + 1)} \]
\[ = \frac{1}{3^n} \]
\[ = \frac{3^2}{3^n} \]
\[ = \frac{3^2}{3^n} \]
Hence, for all \( n > 2 \), we have
\[ 0 \leq \frac{2^n}{n!} \leq \frac{2^n}{2!} \cdot \frac{3^2}{3^n} \]
\[ = \frac{3^2}{2!} \left( \frac{2}{3} \right)^n. \]
Because \( 2/3 < 1 \), we have
\[ \frac{3^2}{2!} \cdot \left( \frac{2}{3} \right)^n \to 0. \]
By the Squeezing Principle, we thus have
\[ \frac{2^n}{n!} \to 0. \]

24.) We claim that if \( \beta \) is any fixed real number, then
\[ \frac{\beta^n}{n!} \to 0. \]
If \( \beta = 0 \), the result is obvious.
If \( \beta \) is a positive integer, you can carry out an argument like the one in the preceding problem, to see that the result is true. (You should do this for practice.)
If \( \beta > 0 \) but not an integer, then we can let \( \alpha \) be any fixed integer bigger than \( \beta \) and since we know, from the previous statement, that
\[ \frac{\alpha^n}{n!} \to 0 \]
and since

\[ 0 \leq \frac{\beta^n}{n!} \leq \frac{\alpha^n}{n!} \]

for all \( n \geq 1 \), we have that \( \frac{\beta^n}{n!} \rightarrow 0 \) by the Squeezing Principle.

Finally, if \( \beta < 0 \), then we note that

\[ -\frac{|\beta|^n}{n!} \leq \frac{\beta^n}{n!} \leq \frac{|\beta|^n}{n!} \]

and since each of the outer sequences both converge to zero, the Squeezing Principle tells us that \( \frac{\beta^n}{n!} \rightarrow 0 \).