Answer to a Question about a Problem Involving the Ratio Test

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One of you asked for a more detailed explanation of Example 3 on page 678 of Grossman so here it is:

We want to apply the Ratio Test to the series

\[ \sum_{k=0}^{\infty} \frac{k^k}{k!}. \]  

(1)

The general term in this series is

\[ a_n = \frac{n^n}{n!}. \]

This gives us

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \]

\[ = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \]

\[ = \frac{(n+1) \cdot (n+1)^n}{n^n} \cdot \frac{n!}{n! \cdot (n+1)} \]

\[ = \frac{n+1}{n+1} \cdot \frac{(n+1)^n}{n^n} \]

\[ = \left( \frac{n+1}{n} \right)^n \]

\[ = \left( 1 + \frac{1}{n} \right)^n \]

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Next, we need to compute
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n. \]

This is a job for L'Hôpital's Rule. (It is a \(1^\infty\) indeterminate form.) We let
\[ y = \left( 1 + \frac{1}{x} \right)^x \]
from which we obtain
\[ \ln y = x \cdot \ln \left( 1 + \frac{1}{x} \right) \]
\[ = \frac{\ln \left( 1 + \frac{1}{x} \right)}{\frac{1}{x}} \]

Now \(\lim_{x \to \infty} \ln y\) is a \(0/0\) indeterminate form so we can use L'Hôpital’s Rule. We have
\[ \lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln \left( 1 + \frac{1}{x} \right)}{\frac{1}{x}} \]
\[ = \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{1}{x}} \left( -\frac{1}{x^2} \right)}{-\frac{1}{x^2}} \]
\[ = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} \]
\[ = 1 \]

Since \(\ln y \to 1\) as \(x \to \infty\), then \(y = e^{\ln y} \to e^1 = e\) as \(x \to \infty\). Thus
\[ \frac{a_{n+1}}{a_n} \to e. \]

Since \(e > 1\), we conclude that the series (1) diverges by the Ratio Test.