1 The Ratio Test

Consider the series
\[ \sum_{k=1}^{\infty} \frac{k^2}{3^k}. \]  

This series has general term
\[ a_n = \frac{n^2}{3^n}. \]

Observe the following:
\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} = \frac{1}{3} \cdot \frac{(n+1)^2}{n^2} = \frac{1}{3} \cdot \left(1 + \frac{1}{n}\right)^2 \]
for all \( n \geq 1 \). Now, for all \( n > 3 \), we have
\[ \frac{a_{n+1}}{a_n} = \frac{1}{3} \cdot \left(1 + \frac{1}{n}\right)^2 < \frac{1}{3} \cdot \left(1 + \frac{1}{3}\right)^2 = \frac{16}{27}. \]

Hence
\[ a_{n+1} < \frac{16}{27} a_n \text{ for all } n > 3. \]

Now, \( a_3 = 1/3 \) so we have
\[ a_4 < \frac{16}{27} \cdot a_3 = \frac{16}{27} \cdot \frac{1}{3} \]
\[ a_5 < \frac{16}{27} \cdot a_4 < \left(\frac{16}{27}\right)^2 \cdot \frac{1}{3} \]
\[ a_6 < \frac{16}{27} \cdot a_5 < \left(\frac{16}{27}\right)^3 \cdot \frac{1}{3} \]
and in general,
\[ 0 < a_n < \left(\frac{16}{27}\right)^{n-3} \cdot \frac{1}{3} = \frac{1}{3} \cdot \left(\frac{27}{16}\right)^3 \cdot \left(\frac{16}{27}\right)^n \]
for all \( n > 4 \). Since the series
\[ \sum_{k=1}^{\infty} \frac{1}{3} \cdot \left(\frac{27}{16}\right)^3 \cdot \left(\frac{16}{27}\right)^k \]
converges (because it is a constant multiple of a convergent geometric series), the series (1) also converges by the Standard Comparison Test.

The above example motivates the Ratio Test.
**Theorem 1 (The Ratio Test)** Let \( \sum_{k=1}^{\infty} a_k \) be a series such that \( a_n > 0 \) for all \( n \) (at least from some point on). Also, suppose that
\[
\frac{a_{n+1}}{a_n} \to L
\]
(where \( L \) is a real number or \( \infty \)).

If \( 0 \leq L < 1 \), then the series \( \sum_{k=1}^{\infty} a_k \) converges and if \( L > 1 \) (including the possibility that \( L = \infty \)), the series \( \sum_{k=1}^{\infty} a_k \) diverges.

**Example 1** Consider the series
\[
\sum_{k=1}^{\infty} \frac{n^{100}}{(1.1)^n}. \tag{2}
\]
We have
\[
\frac{a_{n+1}}{a_n} = \frac{(n + 1)^{100}}{(1.1)^{n+1}} \cdot \frac{(1.1)^n}{n^{100}}
\]
\[
= \frac{1}{1.1} \cdot \left( \frac{n + 1}{n} \right)^{100}
\]
\[
= \frac{10}{11} \cdot \left( 1 + \frac{1}{n} \right)^{100}.
\]
From this, we see that
\[
\frac{a_{n+1}}{a_n} \to \frac{10}{11}
\]
and since \( 10/11 < 1 \), we conclude that the series (2) converges by the Ratio Test.

**Example 2** The Ratio Test does not apply to the harmonic series
\[
\sum_{k=1}^{\infty} \frac{1}{k}
\]
because for this series we have
\[
\frac{a_{n+1}}{a_n} = \frac{n}{n + 1} \to 1.
\]
(Of course, we know from our past work that this series diverges.)
2 Power Series

A power series is a series of the form

\[ b_0 + b_1 (x - x_0) + b_2 (x - x_0)^2 + \cdots \]

where \( x_0 \) is some fixed real number, the coefficients \( b_0, b_1, b_2, \ldots \) are real numbers and \( x \) also stands for a real number but is thought of as a variable.

**Example 3** The series

\[ \sum_{k=0}^{\infty} k (x - 3)^k = 0 + (x - 3) + 2 (x - 3)^2 + 3 (x - 3)^3 + \cdots \]

is a power series.

**Example 4** A Taylor Series

\[ \sum_{k=0}^{\infty} \frac{1}{k!} \cdot f^{(k)}(x_0) \cdot (x - x_0)^k \]

is a power series. A Taylor series is the infinite version of a Taylor polynomial. The coefficients for the Taylor series of a particular function \( f \) are determined (as with Taylor polynomials) by

\[ b_n = \frac{1}{n!} \cdot f^{(n)}(x_0). \]

For example, the Taylor series of \( f(x) = e^x \) at \( x_0 = 0 \) is

\[ \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots \]

3 Convergence of Power Series

Let us examine the power series

\[ b_0 + b_1 (x - x_0) + b_2 (x - x_0)^2 + \cdots \]

with regard to convergence. Recall that \( x_0, b_0, b_1, b_2, \ldots \) are all constants and \( x \) is a variable. We are interested in determining for which values of \( x \)
the power series converges. To begin, it is easy to see that the power series converges when \( x = x_0 \) because in this case, the series is simply

\[
b_0 + 0 + 0 + 0 + \cdots
\]

so the series converges to the sum \( b_0 \). If we attempt to apply the Ratio Test to the power series

\[
\sum_{k=0}^{\infty} b_k (x - x_0)^k,
\]

(assuming that \( b_n \neq 0 \) for any \( n \) and \( x \neq x_0 \)), we obtain

\[
\frac{|b_{n+1} (x - x_0)^{n+1}|}{|b_n (x - x_0)^n|} = \frac{|b_{n+1}|}{|b_n|} \cdot |x - x_0|.
\]

If we know that

\[
\frac{|b_{n+1}|}{|b_n|} \to L \text{ (some finite limit or } \infty),
\]

then we know that the power series converges (absolutely) for all values of \( x \) such that

\[
L \cdot |x - x_0| < 1.
\]

**Example 5** For the power series

\[
\sum_{k=0}^{\infty} \frac{1}{k!} x^k
\]

we have

\[
b_n = \frac{1}{n!}
\]

so

\[
\frac{|b_{n+1}|}{|b_n|} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1} \to 0
\]

which tells us that the series \( 3 \) converges absolutely for all \( x \) such that

\[
0 \cdot |x - 0| < 1.
\]

Since any \( x \in (-\infty, \infty) \) satisfies this, the series converges absolutely for all \( x \in (-\infty, \infty) \).
Example 6  The series
\[ \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \]  \hspace{1cm} (4)
converges absolutely for all \( x \) such that \( |x| < 1 \). Here is why: We have
\[ \left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{1}{n+2} \cdot \frac{n+1}{1} \right| = \frac{n+1}{n+2} \to 1 \]
from which we can see that the series (4) converges absolutely for all values of \( x \) such that
\[ 1 \cdot |x - 0| < 1. \]

Lemma 2  If the power series
\[ \sum_{k=0}^{\infty} b_k (x - x_0)^k \]
converges for some particular value \( x = x_0 + M \) (where \( M \) is some real number), then this same power series converges absolutely for all \( x \) such that \( |x - x_0| < |M| \).

(We will omit the proof of this lemma.)

Theorem 3 (Interval of Convergence)  Given a power series
\[ \sum_{k=0}^{\infty} b_k (x - x_0)^k \]  \hspace{1cm} (5)
there are three possibilities concerning the set of all \( x \) for which this series converges. These possibilities are

1. The series converges only for \( x = x_0 \).

2. The series converges for all \( x \in (-\infty, \infty) \).

3. There is a positive number \( R \) called the radius of convergence of the power series (5) such that the series converges absolutely for all \( x \) such that \( |x - x_0| < R \) and diverges for all \( x \) such that \( |x - x_0| > R \).
Remark 1 In case 1 of the above theorem, we say that the radius of convergence is \( R = 0 \) and in case 2, we say that the radius of convergence is \( R = \infty \).

Definition 1 The interval of convergence, \( I \), of a given power series is defined to be the set of all values, \( x \), such that the power series converges for that \( x \).

Example 7 Let us find the interval of convergence of the power series

\[
\sum_{k=0}^{\infty} \frac{5k^2}{k^2 + 6} (x - 3)^k.
\]

(6)

We have

\[
\left| \frac{b_{n+1}}{b_n} \right| = \frac{5 (n + 1)^2}{(n + 1)^2 + 6} \cdot \frac{n^2 + 6}{5n^2} \to 1
\]

so the series (6) converges for all \( x \) such that \(|x - 3| < 1\) and diverges for all \( x \) such that \(|x - 3| > 1\). The case \(|x - 3| = 1\) must be handled separately. If \(|x - 3| = 1\), then

\[
\left| \frac{5n^2}{n^2 + 6} (x - 3)^n \right| = \left| \frac{5n^2}{n^2 + 6} \right| \to 5
\]

which shows that the series diverges (by the Basic Divergence Test) in this case. Hence, the interval of convergence of (6) is the interval \( I = (2,4) \).

Remark 2 If some of the coefficients, \( b_i \), in the power series

\[
b_0 + b_1 (x - x_0) + b_2 (x - x_0)^2 + \cdots
\]

are zero, we must use the entire term \( b_i (x - x_0)^i \) in applying the Ratio Test to determine the interval of convergence. If we don’t, we might get an incorrect answer for the interval of convergence. The following example illustrates this.

Example 8 Let us find the interval of convergence of the power series

\[
\sum_{k=0}^{\infty} \frac{(2k)^2}{4^k} x^{2k} = \frac{1}{4} x^2 + \frac{1}{16} x^4 + \frac{9}{1024} x^6 + \frac{1}{1024} x^8 + \cdots
\]

Here, we must consider the ratio

\[
\frac{a_{n+1}}{a_n}
\]
where
\[ a_n = \frac{(2n)^2}{4^{2n}} x^{2n} . \]

We have
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2(n+1))^2 x^{2(n+1)}}{4^{2(n+1)}} \cdot \frac{4^{2n}}{(2n)^2 x^{2n}} \right|
= \frac{1}{16} \left( \frac{n+1}{n} \right)^2 x^2
\]
from which we obtain
\[
\left| \frac{a_{n+1}}{a_n} \right| \to \frac{1}{16} x^2.
\]
Hence, this power series converges for all \( x \) such that
\[ \frac{1}{16} x^2 < 1 , \]
that is, for all \( x \) such that \( |x| < 4 \). It is easy to see that the series diverges (by the Basic Divergence Test) at \( x = -4 \) and \( x = 4 \) so the interval of convergence is \( I = (-4, 4) \).

Note that if we had simply considered
\[
\left| \frac{b_{n+1}}{b_n} \right|
\]
where
\[ b_n = \frac{(2n)^2}{4^{2n}} \]
we would have obtained
\[
\left| \frac{b_{n+1}}{b_n} \right| = \frac{1}{16} \left( \frac{n+1}{n} \right)^2 \to \frac{1}{16}
\]
and we would have been incorrect to say that the radius of convergence is 16.