Sequences

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1 What is a Sequence?

Let \( Z^+ \) denote the set of positive integers. That is,

\[ Z^+ = \{1, 2, 3, \ldots \} . \]

A sequence is a function whose domain is \( Z^+ \).

**Example 1** The function defined on \( Z^+ \) be \( f(n) = 1/n^2 \) is a sequence. For this sequence, we have

\[
\begin{align*}
  f(1) & = \frac{1}{1^2} = 1 \\
  f(2) & = \frac{1}{2^2} = \frac{1}{4} \\
  f(3) & = \frac{1}{3^2} = \frac{1}{9} \\
  f(4) & = \frac{1}{4^2} = \frac{1}{16} \\
  \vdots \\
  \text{etc.}
\end{align*}
\]

Instead of using the functional notation, \( f(n) \), it is usually more convenient to use a notation like \( a_n \). For example, for the sequence in the above example, we would write \( a_n = 1/n^2 \).

Although a sequence is technically a function, we usually think of a sequence as an infinite progression of numbers which we call the terms of the
sequence. For example, when we see the sequence \( a_n = \frac{1}{n^2} \), we think of the progression of numbers
\[
1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots
\]
Also, we sometimes find it more convenient to “start” a sequence somewhere other than \( n = 1 \). For example, if \( a_n = \frac{1}{n^2} \) and we write
\[
\{a_n\}_{n=3}^{\infty},
\]
we are referring to the sequence
\[
\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots
\]
Of course, if we want to insist that all sequences start at \( n = 1 \), we could write \( a_n = \frac{1}{(n + 2)^2} \) to obtain the same sequence.

Let us adopt the following convention: If we just write \( a_n \) or \( \{a_n\} \), then we will assume that the sequence is to start at \( n = 1 \), whereas if we write \( \{a_n\}_{n=k}^{\infty} \), we will assume that the sequence is to start at \( n = k \).

**Example 2** The sequence defined by \( a_n = (-1)^n (n + 2) \) is the sequence with terms
\[-3, 4, -5, \ldots\]
whereas the sequence defined by
\[
\{(-1)^n (n + 2)\}_{n=3}^{\infty}
\]
is the sequence with terms
\[8, -9, 10, \ldots\]

## 2 Limits of Sequences

If \( a_n \) is a sequence and if there is a real number \( L \) such that the terms of the sequence are all close to \( L \) for \( n \) sufficiently large, we say that the sequence \( a_n \) converges to the limit \( L \). If no such number \( L \) exists, then we say that the sequence \( a_n \) diverges. Just like with functions which we have previously studied, a divergent sequence may diverge to \( \infty \) or to \( -\infty \) or it may just diverge by oscillation. If the sequence \( a_n \) converges to the limit \( L \), we write
\[
\lim_{n \to \infty} a_n = L
\]

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or we can just simply write
\[ a_n \to L \]
since, for sequences, it is always assumed that we are taking the limit as \( n \to \infty \).

**Example 3** For the sequence \( a_n = 1/n^2 \), the terms are all close to 0 when \( n \) is large, so we have
\[
\lim_{n \to \infty} a_n = 0.
\]
(We say that this sequence converges to 0.)

**Example 4** For the sequence \( a_n = n \), the terms just keep getting bigger and bigger (beyond all bounds) so we have
\[ a_n \to \infty. \]
(We say that this sequence diverges to \( \infty \).)

**Example 5** For the sequence \( a_n = \cos(n\pi) \), the terms of the sequence are
\[ -1, 1, -1, 1, -1, \ldots \]
Since there is no number which all of the terms of the sequence are close to for \( n \) sufficiently large, we say that \( a_n \) diverges. This sequence diverges by oscillation.

### 2.1 Geometric Progressions

A geometric progression is a sequence of the form \( a_n = r^n \) where \( r \) is some fixed real number. An example of a geometric progression is \( a_n = (2/3)^n \). The terms of this particular geometric progression are
\[ \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots \]
If you write out several more terms of this progression, you should notice that the terms are decreasing and getting closer and closer to zero. In fact, each term is \( 2/3 \) of the previous term. Hence we have
\[ \left( \frac{2}{3} \right)^n \to 0. \]

Just using some sample computations, you should be able to convince yourself of the following facts:
1. If $|r| < 1$, then $|r|^n \to 0$.

2. If $|r| > 1$, then $|r|^n \to \infty$

3. If $|r| = 1$, then $|r|^n \to 1$ (This is rather obvious because, in fact, $|r|^n = 1$ for all $n$)

**Example 6** For $r = -4/3$, the sequence

$$\left( -\frac{4}{3} \right)^n$$

diverges by oscillation but note that

$$\left| -\frac{4}{3} \right|^n \to \infty$$

### 2.2 Using L’Hôpital’s Rule to Evaluate Limits of Sequences

Suppose that $a_n$ is a sequence such that there is a function $f$ which is defined for all sufficiently large $x$ and which agrees with $a_n$ when $n$ is an integer, that is $f(n) = a_n$. If $f$ satisfies the hypotheses of L’Hôpital’s Rule, we can use L’Hôpital’s Rule to evaluate the limit of the sequence.

**Example 7** Let us evaluate

$$\lim_{n \to \infty} \frac{n}{e^n}.$$

We do this by noting that if this limit exists, then the value of the limit is the same as the value of the limit

$$\lim_{x \to \infty} \frac{x}{e^x}$$

where, in this latter limit, we are considering the continuous function

$$f(x) = \frac{x}{e^x}.$$

Since this function $f$ satisfies all of the hypotheses of L’Hôpital’s Rule for an $\infty/\infty$ indeterminate form, we have

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

so we can conclude that

$$\lim_{n \to \infty} \frac{n}{e^n} = 0.$$
Caution is in order here. If we have \( f(n) = a_n \) for all \( n \) and
\[
\lim_{x \to \infty} f(x)
\]
does not exist, we cannot conclude that
\[
\lim_{n \to \infty} a_n
\]
does not exist.

Example 8  Consider the sequence
\[
a_n = \sin(\pi n)
\]
and the function
\[
f(x) = \sin(\pi x).
\]
Clearly, if \( n \) is any positive integer, then \( f(n) = a_n \). However, note that
\[
\lim_{x \to \infty} f(x) \text{ does not exist}
\]
but
\[
\lim_{n \to \infty} a_n = 0.
\]

2.3 The Squeezing Principle

Theorem 9 (The Squeezing Principle) Suppose that \( a_n \) and \( b_n \) are sequences which both converge to the limit \( L \) and suppose that \( c_n \) is a sequence such that \( a_n \leq c_n \leq b_n \) for all \( n \) (or at least for all \( n \) from some point on). Then \( c_n \) also converges to the limit \( L \).

Example 10  Let us prove that
\[
\frac{\sin n}{n} \to 0
\]
Since
\[
-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \text{ for all } n = 1, 2, 3, \ldots
\]
and since the sequences \(-1/n\) and \(1/n\) both converge to 0, we have
\[
\frac{\sin n}{n} \to 0.
\]
Example 11 Let us prove that
\[ \frac{97^n}{n!} \to 0. \]

We have
\[ \frac{97^n}{n!} = \frac{97^n \cdot 97!}{97! \cdot n!} = \frac{97^n}{97! \cdot n!}. \tag{1} \]

Now, if \( n = 97 + k \) where \( k \) is a positive integer, we have
\[
\frac{97!}{n!} = \frac{1 \cdot 2 \cdots 96 \cdot 97}{1 \cdot 2 \cdots 96 \cdot 97 \cdot (97 + 1) \cdot (97 + 2) \cdots (97 + k)} = \frac{1}{(97 + 1) \cdot (97 + 2) \cdots (97 + k)} \leq \frac{1}{(97 + 1) \cdot (97 + 1) \cdots (97 + 1)} = \left( \frac{1}{98} \right)^k = \left( \frac{1}{98} \right)^{n - 97} = \frac{98^{97 - n}}{98^n} = \frac{98^{97}}{98^n}.
\]

Returning to (1), we have
\[
\frac{97^n}{n!} = \frac{97^n}{97! \cdot n!} \leq \frac{97^n \cdot 98^{97}}{97! \cdot 98^n} = \frac{98^{97}}{97!} \cdot \left( \frac{97}{98} \right)^n.
\]

Hence, we have that if \( n > 97 \), then
\[ 0 \leq \frac{97^n}{n!} \leq \frac{98^{97}}{97!} \cdot \left( \frac{97}{98} \right)^n. \]
Since \( \frac{97}{98} < 1 \), we have

\[
\left( \frac{97}{98} \right)^n \to 0.
\]

Also, \( \frac{98^{97}}{97!} \) is a constant so

\[
\frac{98^{97}}{97!} \cdot \left( \frac{97}{98} \right)^n \to 0.
\]

We conclude, by the Squeezing Principle, that

\[
\frac{97^n}{n!} \to 0.
\]