Series

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1 Convergence and Divergence of Series

A series is a sum of infinitely many terms

\[ \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \]  

(1)

The sequence \( a_n \) is called the sequence of terms of the series (1) and the sequence

\[ S_n = a_1 + a_2 + \cdots + a_n \]

is called the sequence of partial sums of the series.

Example 1 Consider the series

\[ \sum_{k=1}^{\infty} \frac{1}{k^2}. \]

The sequence of terms of this series is

\[ a_n = \frac{1}{n^2} \]

and the sequence of partial sums of this series is

\[ S_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}. \]
The concept of a series allows us to formalize the idea of adding up infinitely many numbers. In particular, if the sequence of partial sums of a series converges to the real number $L$, then this essentially means that the sum of the infinitely many terms which make up the series is $L$. We take this to be our definition of convergence of a series.

**Definition 2** If $\sum_{k=1}^{\infty} a_k$ is a series with sequence of partial sums $S_n$ and if $S_n$ converges to the limit $L$, then we say that the series converges to $L$ (or that the series has sum $L$) and we write

$$\sum_{k=1}^{\infty} a_k = L$$

If $S_n$ diverges, then we say that the series diverges.

**Example 3 (Geometric Series)** Consider a geometric series

$$\sum_{k=0}^{\infty} r^k.$$  

We have seen that the sequence of partial sums of this series is

$$S_n = 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$  

If $|r| < 1$, then

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{r^{n+1} - 1}{r - 1} = \frac{0 - 1}{r - 1} = \frac{1}{1 - r}.$$  

so in this case, the geometric series converges to the sum $1/(1 - r)$ and we write

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}.$$  

If $|r| \geq 1$, then $S_n$ diverges so, in this case, we say that the geometric series diverges. Note that if $r \geq 1$, then $S_n \to \infty$ so it would be okay in this case to write

$$\sum_{k=0}^{\infty} r^k = \infty$$  

but we still call this series divergent because $\infty$ is not a real number.
Example 4 (The Harmonic Series)  The harmonic series is the series

\[ \sum_{k=1}^{n} \frac{1}{k}. \]

We will show that this series diverges by examining its sequence of partial sums. Note that

\[ S_1 = 1 \]
\[ S_2 = 1 + \frac{1}{2} \]
\[ S_4 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) = 1 + \frac{1}{2} + \frac{1}{2} \]
\[ S_8 = S_4 + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) > 1 + \frac{1}{2} + \frac{1}{2} + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \]

Likewise,

\[ S_{16} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \]

and in general

\[ S_{2^n} > 1 + \frac{1}{2} \cdot n. \]

This shows that the sequence of partial sums is not bounded above and, hence, is divergent. We conclude that the harmonic series diverges.

2 A Basic Criterion for Divergence of a Series

In order for the series

\[ \sum_{k=1}^{\infty} a_k \]

to converge, the terms \( a_n \) must all be very small (in absolute value) for large \( n \). In particular, the following is true:

**Theorem 5** If the series \( \sum_{k=1}^{\infty} a_k \) converges, then \( a_n \to 0 \).

Stated in another way, we have

**Theorem 6 (Basic Divergence Test)** If \( a_n \) does not converge to the limit 0, then the series \( \sum_{k=1}^{\infty} a_k \) diverges.
Example 7  The series
\[ \sum_{k=1}^{\infty} (-1)^n \]
diverges because its sequence of terms is \( a_n = (-1)^n \) and this sequence does not converge to 0.

Example 8  The series
\[ \sum_{k=1}^{n} \frac{k}{k+1} \]
diverges because its sequence of terms is
\[ a_n = \frac{n}{n+1} \]
and this sequence does not converge to 0.

Caution: If it is true that \( a_n \to 0 \), we cannot automatically conclude that the series \( \sum_{k=1}^{n} a_k \) converges! An example of why this is true is the harmonic series
\[ \sum_{k=1}^{\infty} \frac{1}{k} \]
for which we have
\[ a_n = \frac{1}{n} \to 0 \]
but the series still diverges.

3  The Two Basic Questions About Series

Given a series
\[ \sum_{k=1}^{\infty} a_k, \]
there are two basic questions which can be asked:

1. Does the series converge?
2. If the series does converge, then what is its sum?
There are many theorems which provide criteria to determine whether certain types of series converge or diverge. We will study some of these theorems in this course. Once it has been determined that a given series converges, it is usually a much more difficult problem to actually determine the sum of the series. Nonetheless, there are certain classes of series whose sums we can compute. These include geometric series and telescoping series.

### 3.1 Geometric Series

As we have seen, if $|r| < 1$, then the geometric series $\sum_{k=0}^{n} r^k$ converges to the sum $1/(1 - r)$. An interesting application of this fact is that we can express any repeating decimal as a ratio of two integers. For example, the decimal 0.5 is equal to 1/2 and the decimal 0.02 is equal to 2/100 or 1/50 but what about the decimal 0.7333…?

We have

$$0.7333\ldots = \frac{7}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \ldots$$

$$= \frac{7}{10} + \frac{3}{100} \cdot \left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right)$$

$$= \frac{7}{10} + \frac{3}{100} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k$$

$$= \frac{7}{10} + \frac{3}{100} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{7}{10} + \frac{3}{10 - 1}$$

$$= \frac{7}{10} + \frac{3}{90}$$

$$= \frac{63}{90} + \frac{3}{90}$$

$$= \frac{66}{90}$$

$$= \frac{11}{15}$$

Thus $0.7333\ldots = \frac{11}{15}$. 

5
3.2 Telescoping Series

An example of a telescoping series is

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)}. \]

Note that

\[ \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \]

Hence, if we write out the sequence of partial sums of this series, we obtain

\[ S_1 = 1 - \frac{1}{2} \]
\[ S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \]
\[ S_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \]
\[ \vdots \]
\[ \text{etc.} \]

In general, we have

\[ S_n = 1 - \frac{1}{n+1} \]

so \( S_n \to 1 \). We conclude that

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1. \]

4 Convergence/Divergence Tests for Series with Non-negative terms

Here we consider two convergence/divergence tests which apply to series with non-negative terms.

4.1 The Integral Test

Theorem 9 (The Integral Test) Let \( \sum_{k=1}^{n} a_k \) be a series such that \( a_n \geq 0 \) for all \( n \geq 1 \) and suppose that there is a function \( f \) such that \( f \) is decreasing
on some interval \([N, \infty)\) and \(f (n) = a_n\) for all \(n \geq N\). Then the series 
\[\sum_{k=1}^{n} a_k\] converges if and only if the improper integral 
\[\int_{N}^{\infty} f (x) \, dx\] converges.

**Proof.** (will be discussed in class)

**Example 10 (p series)** A p series is a series of the form 
\[\sum_{k=1}^{\infty} \frac{1}{k^p}\] where \(p\) is some fixed positive real number. We will show that the p series converges if \(p > 1\) and diverges if \(0 < p \leq 1\). First, let us assume that \(p \neq 1\). If we let \(f (x) = 1/x^p\), then \(f\) is decreasing on \([1, \infty)\) and \(f (n) = 1/n^p\) for \(n \geq 1\). Also,
\[
\int_{1}^{\infty} f (x) \, dx = \int_{1}^{\infty} \frac{1}{x^p} \, dx \\
= \lim_{N \to \infty} \int_{1}^{N} \frac{1}{x^p} \, dx \\
= \lim_{N \to \infty} \int x^{-p} \, dx \\
= \lim_{N \to \infty} \left[ \frac{1}{1-p} \cdot x^{-p+1} \right]_{1}^{N} \\
= \lim_{N \to \infty} \frac{1}{1-p} \cdot \frac{1}{N^{p-1}} - 1
\]

If \(p > 1\), then
\[
\lim_{N \to \infty} \frac{1}{1-p} \cdot \frac{1}{N^{p-1}} - 1 = \frac{1}{1-p}
\]
which shows that, in this case, the improper integral 
\[\int_{1}^{\infty} f (x) \, dx\] converges. If \(0 < p < 1\), then
\[
\lim_{N \to \infty} \frac{1}{1-p} \cdot \frac{1}{N^{p-1}} - 1 = \infty
\]
which shows that, in this case, the improper integral
\[ \int_1^\infty f(x) \, dx \]
diverges. By the Integral Test, we conclude that the \( p \) series converges if \( p > 1 \) and diverges if \( 0 < p < 1 \).

If \( p = 1 \), then we have the harmonic series which we have already shown diverges. As an exercise, use the Integral Test to show that the harmonic series diverges.

### 4.2 The Standard Comparison Test

**Theorem 11 (The Standard Comparison Test)** Suppose that \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) are series such that \( 0 \leq a_n \leq b_n \) for all \( n \geq 1 \) and such that \( \sum_{k=1}^{\infty} b_k \) converges. Then \( \sum_{k=1}^{\infty} a_k \) converges.

**Remark 1** The Standard Comparison Test can also be used to obtain divergence. If the series \( \sum_{k=1}^{\infty} a_k \) in the above theorem diverges, it must be the case that \( \sum_{k=1}^{\infty} b_k \) diverges.

**Example 12** Let us show that the series
\[ \sum_{k=1}^{\infty} \frac{1}{k^2 + k + 18} \]
converges.

Since
\[ 0 \leq \frac{1}{n^2 + n + 18} \leq \frac{1}{n^2} \]
for all \( n \geq 1 \) and since
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \]
is a convergent \( p \) series, then
\[ \sum_{k=1}^{\infty} \frac{1}{k^2 + k + 18} \]
converges by the Standard Comparison Test.