Trigonometric Substitution
Trigonometric Substitution can be used to handle certain integrals whose integrands contain $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a$ is a constant.

The strategy is:

1.) If the integrand contains $\sqrt{a^2 - x^2}$, then make the substitution $x = a \sin(\theta)$.

2.) If the integrand contains $\sqrt{a^2 + x^2}$, then make the substitution $x = a \tan(\theta)$.

1.) If the integrand contains $\sqrt{x^2 - a^2}$, then make the substitution $x = a \sec(\theta)$.
Example 1

Set up a definite integral that gives the area of a circle of radius $R$ (where $R$ is some given positive number) and then use trigonometric substitution to evaluate this integral.
Exercise 1

An ellipse has an equation of the form
\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1
\]
where \(a\) and \(b\) are positive constants. As an example, the ellipse
\[
\left( \frac{x}{2} \right)^2 + \left( \frac{y}{4} \right)^2 = 1
\]
is pictured below.

By using integration, show that the area of the ellipse described by \((x/a)^2 + (y/b)^2 = 1\) is \(\pi ab\).
Example 2

\[ \int \frac{1}{4+x^2} \, dx = ? \]
Two Integrals to Add to Your Basic Toolbox of Integrals

\[
\int \sec(\theta) \, d\theta = \ln|\sec(\theta) + \tan(\theta)| + C
\]
\[
\int \csc(\theta) \, d\theta = -\ln|\csc(\theta) + \cot(\theta)| + C
\]

To verify that the above two integration formulas are correct, we use the fact (which is also useful in many other situations) that

\[
\frac{d}{du}(|u|) = \frac{|u|}{u}.
\]
Example 3

\[ \int \frac{1}{\sqrt{4+x^2}} \, dx = ? \]
Example 4

\[ \int \frac{1}{\sqrt{x^2 - 1}} \, dx = ? \]
To become good at integration requires practice. One must carefully look at each problem and decide what the best approach is. The next example gives an integral with an integrand that contains $\sqrt{1 - x^2}$ but for which trigonometric substitution is not necessary.
Example 5

\[ \int x \sqrt{1 - x^2} \, dx = ? \]