Instructions. Show all of your work! You will not get full credit if you don’t include correct notation. In particular, you must write “=” where needed.

1. (a) Sketch the parametric curve

\[ x = 5 \sin(t) \]
\[ y = 2 \cos(t) \]
\[ 0 \leq t \leq 2\pi. \]

Indicate with an arrow (on your sketch) the direction in which the curve is traced out as the parameter increases.

Answer: The curve, whose graph is shown here, is an ellipse. It is traced out clockwise.

(b) Find a Cartesian equation for this curve by eliminating the parameter \( t \).

Solution: Since

\[ \frac{x}{5} = \sin(t) \]
\[ \frac{y}{2} = \cos(t), \]

we have

\[ \frac{x^2}{25} + \frac{y^2}{4} = 1. \]

2. Find the area of the region bounded by the curves \( y = x^2 \) and \( y = 4x - x^2 \).

Solution: The region is pictured below. The extremities are the points (0, 0) and (2, 4). The curve \( y = x^2 \) is on the bottom.
The area of the region is
\[
A = \int_{0}^{2} \left( (4x - x^2) - x^2 \right) \, dx \\
= \int_{0}^{2} (4x - 2x^2) \, dx \\
= \frac{8}{3}.
\]

3. Find the volume of the solid obtained by revolving the region bounded by the curves \( y = x \) and \( y = \sqrt{x} \) about the line \( y = 1 \).

**Solution:** The region is pictured below.
Using the slab method, we obtain

\[ V = \int_0^1 \pi \left( (1 - x)^2 - (1 - \sqrt{x})^2 \right) \, dx = \frac{\pi}{6}. \]

Using the shell method, we obtain

\[ V = \int_0^1 2\pi (1 - y) (y - y^2) \, dy = \frac{\pi}{6}. \]

4. Find the volume of a cap of a sphere of height \( h \) that sits atop a sphere of radius \( R \). (It’s the details that count!)

\[ \text{Solution: } \text{This was done in class.} \]

5. The graph of the parametrically defined curve

\[ \begin{align*}
    x &= e^t - t \\
    y &= 4e^{\frac{t}{2}} \\
    -8 &\leq t \leq 3
\end{align*} \]

is shown below. Find the length of this curve. (The answer is approximately equal to 31, but you should find the exact answer expressed in terms of the number \( e \).)
Solution: Since
\[
\frac{dx}{dt} = e^t - 1 \\
\frac{dy}{dt} = 2e^{\frac{t}{2}},
\]
we have
\[
\left(\frac{dx}{dt}\right)^2 = (e^t - 1)^2 = e^{2t} - 2e^t + 1 \\
\left(\frac{dy}{dt}\right)^2 = \left(2e^{\frac{t}{2}}\right)^2 = 4e^t.
\]
Thus
\[
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2.
\]
Therefore the arc length is
\[
\int_{-8}^{3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{-8}^{3} (e^t + 1) \, dt = e^3 - e^{-8} + 11 \approx 31.085.
\]