Instructions. Show all of your work! You will not get full credit if you don’t include correct notation. In particular, you must write “=” where needed. Also, when discussing sequences and series, take care not to write the word “it”.

1. Does L’Hospital’s Rule apply to the problem

\[ \lim_{x \to 1} \frac{e^x}{x^2} \]

If so, then use L’Hospital’s Rule to find the value of this limit. If not, then explain why not and find the correct value of the limit.

\textbf{Answer:} Since \( \lim_{x \to 1} x^2 = \infty \), L’Hospital’s Rule does apply. We see that

\[ \lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} \]

(assuming that the limit on the right exists or is \( \infty \)). L’Hospital’s Rule also applies to the limit on the right and we obtain

\[ \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty. \]

Thus

\[ \lim_{x \to \infty} \frac{e^x}{x^2} = \infty. \]

2. Decide whether the sequence

\[ a_n = \frac{(-1)^n \cdot n^3}{n^3 + 2n^2 + 1} \]

is convergent or divergent. Explain your reasoning. Write in complete sentences that do not contain the word “it”.

\textbf{Answer:} This sequence is divergent. To see why, note that

\[ \lim_{n \to \infty} \frac{n^3}{n^3 + 2n^2 + 1} \quad \lim_{n \to \infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} = \frac{1}{1 + 0 + 0} = 1. \]

Thus, if we did not have the \((-1)^n\), then we would have a sequence that converges to 1. However, because of the \((-1)^n\), the terms of \(a_n\) alternate in sign and thus, as \(n \to \infty\), the terms of \(a_n\) bounce back and forth between 1 and \(-1\). This means that \(a_n\) diverges by oscillation.

3. Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{2n}{3n + 1} \]
is convergent or divergent. If the series is convergent, then find its sum. If the series is divergent, then explain why (writing in complete sentences that do not contain the word “it”).

**Answer:** The Basic Divergence Test tells us that this series is divergent because

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n}{3n + 1} = \frac{2}{3} \neq 0.
\]

4. Determine whether the improper integral

\[ \int_0^\infty \cos(x) \, dx \]

is convergent or divergent. If this integral is convergent, then determine its value.

**Solution:** This integral is divergent. Note that for any \( t > 0 \) we have

\[
\int_0^t \cos(x) \, dx = \sin(x)|_0^t = \sin(t) - \sin(0) = \sin(t).
\]

However, \( \lim_{t \to \infty} \sin(t) \) does not exist. Therefore the integral is divergent.

5. Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{1 + \sin(n)}{10^n} \]

is convergent or divergent. You must explain your reasoning by referring to one or more of the “tests” we have studied (Basic Divergent Test, Integral Comparison Test, Standard Comparison Test, Limit Comparison Test).

**Solution:** Note that \( -1 \leq \sin(n) \leq 1 \) for all \( n \geq 1 \). By adding 1 to all parts of this inequality, we obtain that \( 0 \leq 1 + \sin(n) \leq 2 \) for all \( n \geq 1 \). Now we can divide all parts of the preceding inequality by \( 10^n \) to obtain

\[
0 \leq \frac{1 + \sin(n)}{10^n} \leq \frac{2}{10^n}
\]

for all \( n \geq 1 \).

Since \( \sum_{n=1}^{\infty} \frac{1}{10^n} \) is a convergent series (because it is a geometric series with \( r = 1/10 \)), then the series \( \sum_{n=1}^{\infty} \frac{2}{10^n} \) is also convergent (because its terms are 2 times the terms of the previously-mentioned convergent geometric series). The SCT now allows us to conclude that the series

\[ \sum_{n=1}^{\infty} \frac{1 + \sin(n)}{10^n} \]

is convergent.

6. Explain why the series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n^3} \]
converges. Your explanation should refer to either the Absolute Convergence Test (ACT) or the Alternating Series Test (AST). (Write in complete sentences that do not contain the word “it”.)

**Answer:** We are considering the series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \frac{(-1)^{n+1}}{n^3}.$$ 

Since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent (because it is a $p$ series with $p = 3$), then we can conclude by the ACT that $\sum a_n$ also converges.