1. Try to use the Ratio Test to determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n}$$

converges or diverges. Does the Ratio Test succeed or fail? If the test succeeds, then does it tell you that the series converges or diverges (and why)? If the test fails, then why does it fail?

Solution: Since

$$a_n = \frac{\sqrt{n}}{1+n},$$

then

$$a_{n+1} = \frac{\sqrt{n+1}}{1+(n+1)}$$

and

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1}}{1+(n+1)} \cdot \frac{1+n}{\sqrt{n}} = \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}.$$  

Since

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2} = \sqrt{1} \cdot 1 = 1,$$

the Ratio Test fails to give any information about the convergence or divergence of the series. (The series can be shown to diverge by the Limit Comparison Test.)

2. Find the radius of convergence of the power series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln(n)}.$$  

Solution: Note that

$$b_n = (-1)^n \frac{x^n}{4^n \ln(n)}$$

and

$$b_{n+1} = (-1)^{n+1} \frac{x^{n+1}}{4^{n+1} \ln(n+1)}$$
and hence
\[
\left| \frac{b_{n+1}}{b_n} \right| = \frac{4^n \ln(n)}{4^{n+1} \ln(n+1)} |x| = \frac{1}{4} \cdot \frac{\ln(n)}{\ln(n+1)} \cdot |x|.
\]
Since (by L.R.)
\[
\lim_{t \to \infty} \frac{\ln(t)}{\ln(t+1)} = \lim_{t \to \infty} \frac{\frac{1}{t}}{\frac{1}{t+1}} = \lim_{t \to \infty} \frac{t+1}{t} = 1,
\]
we see that
\[
\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \frac{1}{4} \cdot \frac{\ln(n)}{\ln(n+1)} \cdot |x| = \frac{1}{4} |x|.
\]
By the Ratio Test, we conclude that the given power series converges for all \( x \) such that \( \frac{1}{4} |x| < 1 \); in other words, for all \( x \) such that \( |x| < 4 \). Hence the radius of convergence is 4.

3. Begin with the known power series representation
\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots
\]
which holds for all \( x \in (-1, 1) \) and use substitutions and algebra to obtain the power series representation of
\[
\frac{x}{1 + 4x}.
\]
What is the interval of convergence of the power series that you obtain? (Be detailed in your solution.)

**Solution:** Note that
\[
x \left( \frac{1}{1-x} \right) = x \cdot \frac{1}{1 - (-4x)}.
\]
Since
\[
\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty} x^n \quad \text{for all } x \text{ such that } |x| < 1,
\]
then
\[
\frac{1}{1 - (-4x)} = 1 + \sum_{n=1}^{\infty} (-4x)^n \quad \text{for all } x \text{ such that } |-4x| < 1.
\]
The inequality \( |-4x| < 1 \) is equivalent to \( |x| < 1/4 \). The radius of convergence of the latter series is thus 1/4. Finally
\[
\frac{x}{1 + 4x} = x \left( 1 + \sum_{n=1}^{\infty} (-4x)^n \right) = x + \sum_{n=1}^{\infty} (-4x)^n x
\]
and this series also has radius of convergence of 1/4. Some other ways to write this series (with and without summation notation) are
\[
\frac{x}{1+4x} = x + \sum_{n=1}^{\infty} (-4)^n x^{n+1} = x - 4x^2 + 4^2 x^3 - 4^3 x^4 + \cdots.
\]
4. Find the Taylor Series of \( f(x) = x^3 - 4x^2 - 4x - 1 \) centered at \( a = -1 \). Show all details of your work.

**Solution:** Since

\[
\begin{align*}
    f^{(0)}(x) &= x^3 - 4x^2 - 4x - 1 \\
    f^{(1)}(x) &= 3x^2 - 8x - 4 \\
    f^{(2)}(x) &= 6x - 8 \\
    f^{(3)}(x) &= 6
\end{align*}
\]

and \( f^{(n)}(x) = 0 \) for all \( n \geq 4 \), we obtain

\[
\begin{align*}
    a_0 &= \frac{f^{(0)}(-1)}{0!} = -2 \\
    a_1 &= \frac{f^{(1)}(-1)}{1!} = 7 \\
    a_2 &= \frac{f^{(2)}(-1)}{2!} = -7 \\
    a_3 &= \frac{f^{(3)}(-1)}{3!} = 1
\end{align*}
\]

and \( a_n = 0 \) for all \( n \geq 4 \). Hence

\[
    f(x) = -2 + 7(x + 1) - 7(x + 1)^2 + (x + 1)^3.
\]

5. Use the (known) MacClaurin series for \( \cos(x) \) to obtain the MacClaurin series for \( \cos(x)/x \) and then obtain an approximation of the definite integral

\[
\int_{1/2}^{1} \frac{\cos(x)}{x} \, dx.
\]

(In obtaining your approximation of the integral, use only the first three terms of the series. The answer that I get is approximately 0.51541.)

**Solution:** Since

\[
\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \text{ for all } x,
\]

then

\[
\frac{\cos(x)}{x} = \frac{1}{x} - \frac{1}{2}x + \frac{1}{24}x^3 - \cdots \text{ for all } x \neq 0
\]

and we obtain

\[
\begin{align*}
\int_{1/2}^{1} \frac{\cos(x)}{x} \, dx \\
&\approx \ln(x) - \frac{1}{4}x^2 + \frac{1}{96}x^4 \bigg|_{x=1/2}^{x=1} \\
&= \left( \ln(1) - \frac{1}{4}(1)^2 + \frac{1}{96}(1)^4 \right) - \left( \ln \left( \frac{1}{2} \right) - \frac{1}{4} \left( \frac{1}{2} \right)^2 + \frac{1}{96} \left( \frac{1}{2} \right)^4 \right) \\
&\approx 0.51541.
\end{align*}
\]
6. Since

\[ \lim_{x \to 0} \frac{\sin(x)}{x} \]

is a limit problem of the indeterminate form \(0/0\), we can use L’Hospital’s Rule to find that

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1. \]

Another way to show that

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \]

is to use the MacClaurin series of \(\sin(x)\). Show how this is done.

**Solution:** Since

\[ \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \text{ for all } x, \]

then

\[ \frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \text{ for all } x \neq 0. \]

Since power series are continuous functions, we obtain

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \left( 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \right) \]
\[ = 1 - 0 + 0 - 0 + \cdots \]
\[ = 1. \]