A graph of the curve with parametric equation
\[ x = 3t - t^3 \]
\[ y = 3t^2 \]
is shown below. (The graph that is shown is traced out as \( t \) varies from \(-2\) to \(2\)).

Use integration to find the arc length of the loop of this curve. In order to receive full credit, your solution must include

- how you determine the \( t \) values where the loop starts and ends.
- details of how you set up the right definite integral that is needed to find the arc length of the loop
- evaluation of the definite integral (including details).
Solution: By looking at the graph, it appears that the loop starts and ends at the point $(0, 9)$. Setting $y = 9$, we obtain $3t^2 = 9$ which gives $t = \pm \sqrt{3}$. Plugging either $t = \sqrt{3}$ or $t = -\sqrt{3}$ into the equation $x = 3t - t^3$ gives $x = 0$. Thus we are correct in stating that the loop starts and ends at the point $(0, 9)$ and we also know that this point corresponds to $t = -\sqrt{3}$ and $t = \sqrt{3}$.

Now note that

\[
\frac{dx}{dt} = 3 - 3t^2 \\
\frac{dy}{dt} = 6t
\]

and so

\[
\left(\frac{dx}{dt}\right)^2 = (3 - 3t^2)^2 = 9 - 18t^2 + 9t^4 \\
\left(\frac{dy}{dt}\right)^2 = (6t)^2 = 36t^2
\]

which gives

\[
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9 + 18t^2 + 9t^4 = (3 + 3t^2)^2
\]

and hence

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3 + 3t^2.
\]

The arc length of the loop is thus

\[
\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2) dt
\]

\[
= 3 \int_{-\sqrt{3}}^{\sqrt{3}} (1 + t^2) dt
\]

\[
= 3 \left[ t + \frac{1}{3}t^3 \right]_{-\sqrt{3}}^{\sqrt{3}}
\]

\[
= 3 \left( \left( \sqrt{3} + \frac{1}{3}(\sqrt{3})^3 \right) - \left( -\sqrt{3} + \frac{1}{3}(-\sqrt{3})^3 \right) \right)
\]

\[
= 12\sqrt{3}.
\]