1. Explain why the series
\[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \]
converges.

**Answer:** Let \( u_n = \frac{1}{n \ln(n)} \). Then

- \( u_n > 0 \) for all \( n \geq 2 \) (because \( \ln(n) > 0 \) for all \( n \geq 2 \)).
- \( u_{n+1} < u_n \) for all \( n \geq 2 \) (because \( \ln(n+1) > \ln(n) \) for all \( n \geq 2 \)).
- \( \lim_{n \to \infty} u_n = 0 \) (because \( \lim_{n \to \infty} \ln(n) = \infty \)).

Therefore the series \( \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)} \) converges by the Alternating Series Test.

2. Is the series
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{100^n}{n!} \]
absolutely convergent, conditionally convergent, or divergent. Explain your answer.

(No credit without correct explanation.)

**Answer:** Let \( a_n = (-1)^{n+1} \frac{100^n}{n!} \). Then

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \frac{100}{n+1}. \]

Since \( \lim_{n \to \infty} \frac{100}{n+1} = 0 < 1 \), the series \( \sum |a_n| \) converges by the Ratio Test. Therefore the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

3. Find the radius of convergence and the interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{x^n}{n (\ln(n))^2}. \]

(You can get up to 10 points on this problem if you just find the radius of convergence.)

**Solution:** Let \( u_n = \frac{1}{n (\ln(n))^2} x^n \). Then

\[ \left| \frac{u_{n+1}}{u_n} \right| = \frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n (\ln(n))^2}{|x|^n} \]

\[ = \frac{n}{n+1} \left( \frac{\ln(n)}{\ln(n+1)} \right)^2 |x|. \]
From this we see that
\[ \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = |x| \]
and the Ratio Test tells us that the power series \( \sum_{n=1}^{\infty} \frac{x^n}{n (\ln(n))^2} \) converges absolutely for all values of \( x \) such that \( |x| < 1 \) (and diverges for all values of \( x \) such that \( |x| > 1 \)). Thus the radius of convergence is \( R = 1 \).

We will now test the series for convergence/divergence at the endpoints \( x = 1 \) and \( x = -1 \).

At \( x = 1 \) we have the series
\[ \sum_{n=1}^{\infty} \frac{1}{n (\ln(n))^2} \]
which converges by the Integral Test. At \( x = -1 \) we have the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n (\ln(n))^2} \]
which is absolutely convergent (for the reason given for the convergence of the series in the \( x = 1 \) case). Therefore the interval of convergence of the given power series is \( I = [-1, 1] \).

4. Show that the MacClaurin series generated by the function \( f(x) = \ln (1 + x) \) is
\[ x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots . \]
(There is more than one way to do this problem. Choose whichever way you like.)

**Solution:** This was done in class. You can either do it by computing the coefficients
\[ c_n = \frac{f^{(n)}(0)}{n!} \]
or by beginning with the known MacClaurin series for \( \frac{1}{1-x} \) and making a substitution and then performing term–by–term integration.

5. a) Use a MacClaurin polynomial of degree 5 to estimate the value of
\[ \int_0^1 \ln (1 + x) \, dx. \]
(\textit{Hint:} You can use what is given in problem 4.)

b) Use integration techniques studied in this course to find the exact value of the above integral.
(You can get up to 16 points credit on this question if you do either part a or part b completely correctly – but of course you must show all of your work!)

**Solution:** a) We have
\[ P_5(x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \frac{1}{5} x^5. \]
Thus
\[
\int_0^1 P_5(x) \, dx = \left. \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{12} x^4 - \frac{1}{20} x^5 + \frac{1}{30} x^6 \right|_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} + \frac{1}{30} = \frac{2}{5}.
\]

b) To evaluate the integral exactly, we first let \( z = x + 1 \) (and hence \( dz = dx \)) to obtain
\[
\int \ln (1 + x) \, dx = \int \ln (z) \, dz
\]
and then we use integration by parts with
\[
\begin{align*}
  u &= \ln (z) & v &= z \\
  du &= \frac{1}{z} \, dz & dv &= dz
\end{align*}
\]
to obtain
\[
\int \ln (z) \, dz = uv - \int v \, du = z \ln (z) - \int 1 \, dz = z \ln (z) - z + C
\]
Thus
\[
\int \ln (1 + x) \, dx = \int \ln (z) \, dz = z \ln (z) - z + C = (x + 1) \ln (x + 1) - x - 1 + C.
\]
Note that the constant 1 can be “absorbed” into the integration constant (that is, we could let \( D = -1 + C \)). Thus
\[
\int \ln (1 + x) \, dx = (x + 1) \ln (x + 1) - x + D.
\]
Now, by the Fundamental Theorem of Calculus, we obtain
\[
\int_0^1 \ln (1 + x) \, dx = (x + 1) \ln (x + 1) - x \bigg|_{x=0}^{x=1} = 2 \ln (2) - 1.
\]