Instructions. Your solution to each problem must contain sufficient detail so that it is clear to the reader what your reasoning process is. Full credit will not be given for solutions that are lacking in detail (such as omission of important steps in computing an integral, etc.) State your conclusions and other important steps in your reasoning in complete sentences. You may use a calculator on this exam but you may not use any books or notes. Note: This exam contains five problems, but you only need to do four of them (of your choice). Even if you work on more than four problems, I will only grade four of them. You must choose (by circling below) the four problems that you want me to grade.

1. Decide which of the functions given in parts a–d are solutions of the differential equation

\[ \frac{dy}{dt} = e^t + y - 1. \]

Note: You are not being asked to solve this differential equation. You are just being asked to decide which of the four functions below are solutions. You must show your procedure of checking to see whether these functions are solutions. Also state your conclusions in complete sentence form: “This function is a solution” or “This function is not a solution”.

(a) \( y = te^t + 1 \)
(b) \( y = (t - 2)e^t + 1 \)
(c) \( y = (t + 4)e^t \)
(d) \( y = te^t - 2 \)

2. Consider the separable differential equation

\[ \frac{dy}{dt} = ty. \]

(a) Find the general solution of this differential equation (including all steps in your reasoning process).
(b) Show how to check to see that the general solution you obtained in part a is correct. Is it correct?
(c) Find the particular solution of this differential equation that satisfies the condition \( y(0) = 3 \).
3. A tank contains 30 gallons of saltwater with a concentration of 0.5 pounds of salt per gallon. There is a stirrer in the tank that keeps the saltwater well–mixed at all times. Saltwater with a concentration of 0.1 pounds of salt per gallon is added to the tank at the rate of 2 gallons per minute while the saltwater mixture is allowed to drain from the tank at the same rate.

(a) Explain, without doing any detailed mathematics, why the concentration of the saltwater in the tank should decrease over time. What should happen to the concentration of the saltwater in the tank as $t \to \infty$?

(b) Let $y(t)$ be the amount (in pounds) of salt in the tank at time $t$ and let $c(t)$ be the concentration (in pounds per gallon) of salt in the tank at time $t$. Find formulas for $y(t)$ and $c(t)$. (To do this, you will have to set up and solve an initial value problem for $y(t)$. Once you find $y(t)$, then it is easy to find $c(t)$.)

(c) For the function, $c(t)$, that you found in part b, evaluate $\lim_{t \to \infty} c(t)$. Does this value agree with your intuitive answer from part a?

4. A thermometer is taken from a room where the temperature is 25$^\circ$C to the outdoors where the temperature is 5$^\circ$C. After one minute, the thermometer reads 15$^\circ$C.

(a) By using Newton’s Law of Cooling (and setting up and solving an appropriate initial value problem), find the function $T(t)$ that gives the temperature of the thermometer at time $t$.

(b) After how long will the temperature of the thermometer reach 7$^\circ$C?

5. Suppose that an object of mass $m$ falls through the atmosphere (near the earth). Assuming no drag force (an unrealistic situation, but one that can be obtained in a vacuum), this situation is modelled by the equation

$$ma = -mg$$

where $g \approx 9.8 \text{ m/s}^2$ is the acceleration of the object due to gravity, $v = v(t)$ is the velocity of the object at time $t$, and $a = a(t)$ is the acceleration of the object at time $t$. Assuming that the initial velocity of the object is $v(0) = v_0$, set up and solve an initial value problem that gives us the function $v(t)$. (Hint: Recall that $a = \frac{dv}{dt}$.) After finding $v(t)$, and assuming that the initial displacement of the object (from ground level) is $y(0) = y_0$, set up and solve an initial value problem that gives us $y(t)$ (the displacement of the object at time $t$).