Instructions. Your solution to each problem must contain sufficient detail so that it is clear to the reader what your reasoning process is. Full credit will not be given for solutions that are lacking in detail (such as omission of important logical steps in your reasoning). State your conclusions and other important steps in your reasoning in complete sentences. Do not use the word “it” anywhere in your writing. You may use a calculator on this exam but you may not use any books or notes. Also, throughout this exam, you may assume knowledge of the convergence/divergence behavior of geometric series and $p$ series and you may also assume knowledge of the Maclaurin series’ of $\frac{1}{1-x}$, $e^x$, $\sin(x)$, $\cos(x)$, and $\arctan(x)$.

Finally, this exam contains six questions but you are only required to do five of them (any five of your choice). Even if you work on all six questions, I will grade only five of them. You must choose which one you do not want me to grade by placing an X over the one that you do not want me to grade in the table below.

1 2 3 4 5 6
1. Write in complete sentences. Do not use the word “it”.

(a) Explain what it means for a series, $\sum_{n=0}^{\infty} a_n$, to be absolutely convergent.

(b) Explain why the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$$

is absolutely convergent.

(c) Explain why the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

is not absolutely convergent.
2. Use the Alternating Series Test to explain why the series

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}} \]

is convergent. (Write in complete sentences. Do not use the word “it”. In your explanation, make sure to verify that all of the hypotheses of the Alternating Series Test are satisfied.)
3. Use the Ratio Test to determine whether the series

\[ \sum_{n=1}^{\infty} \frac{2^n}{n^3} \]

converges or diverges. (Write in complete sentences. Do not use the word “it”.)
4. Suppose that

\[
\sum_{n=0}^{\infty} a_n (x+3)^n = a_0 + a_1 (x+3) + a_2 (x+3)^2 + a_4 (x+3)^4 + \cdots
\]

is a power series and suppose that we know that this power series converges when \(x = 2\). Decide whether each of the following statements must be true or is not necessarily true. (Circle the correct choice.)

(a) This power series converges when \(x = 3\).
   (must be true / is not necessarily true).

(b) This power series converges when \(x = -8\).
   (must be true / is not necessarily true).

(c) This power series converges when \(x = 0\).
   (must be true / is not necessarily true).

(d) This power series converges when \(x = -4\).
   (must be true / is not necessarily true).
(a) Construct the Taylor series, centered at $a = 9$, for the function

$$f(x) = \frac{1}{\sqrt{x}}.$$  

(Be sure to include all details of your work. You can earn up to 15 points if you do this part correctly – even if part b is not done correctly.)

(b) Find the radius of convergence ($R$) of the Taylor series that you constructed in part a. (Circle the correct choice below. However, you will not credit unless you provide correct reasoning.)

1. $R = 9$
2. $R = 1/2$
3. $R = 1/3$
4. $R = 1/9$
5. $R = \infty$
6. none of the above
5. Use the Maclaurin Series (which you should know) for $\arctan(x)$ to obtain the Maclaurin series for $\arctan(x^2)$. Then, by observing that

$$\frac{d}{dx} \left( \arctan(x^2) \right) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

use term–by–term differentiation to express

$$\frac{2x}{1+x^4}$$

as a power series.

**Report your results here:**

(a) The Maclaurin series of $\arctan(x)$ is

$$\arctan(x) = \frac{1}{1+x^2} = \frac{2x}{1+x^4}$$

(b) The Maclaurin series of $\arctan(x^2)$ is

$$\arctan(x^2) = \frac{2x}{1+x^4}$$

(c) The Maclaurin series of $2x/(1+x^4)$ is

$$\frac{2x}{1+x^4} = \frac{2x}{1+x^4}$$