**Instructions.** Your solution to each problem must contain sufficient detail so that it is clear to the reader what your reasoning process is. Full credit will not be given for solutions that are lacking in detail (such as omission of important steps in computing an integral, etc.) State your conclusions and other important steps in your reasoning in complete sentences. You may use a calculator on this exam but you may not use any books or notes. **Note:** This exam contains five problems, but you only need to do four of them (of your choice). Even if you work on more than four problems, I will only grade four of them. You must choose (by circling below) the four problems that you want me to grade.

![Circle choices: 1, 2, 3, 4, 5](#)

1. **Only one** of the functions given in parts a–e is a solution of the differential equation

\[
\frac{dy}{dt} = y + t.
\]

Decide which one it is and show that it is.

**Note:** You are **not** being asked to solve the above differential equation. You are just being asked to decide which one of the five functions given below is a solution. You must decide which one and you must show the verification (check).

(a) \( y = t + 1 \)
(b) \( y = 3e^t - t \)
(c) \( y = 3e^t - t - 1 \)
(d) \( y = 3e^t + 4t - 1 \)
(e) \( y = 3e^t - t - 2 \)

**Answer:** The function \( y = 3e^t - t - 1 \) is a solution of the given differential equation.

**Show the verification of your result here:**

Let \( y = 3e^t - t - 1 \). Then

\[
\frac{dy}{dt} = 3e^t - 1
\]

and

\[
y + t = (3e^t - t - 1) + t = 3e^t - 1.
\]

Thus

\[
\frac{dy}{dt} = y + t.
\]

2. Consider the separable differential equation

\[
\frac{dy}{dt} = 3 - y.
\]
(a) Find the general solution of this differential equation (including all steps in your reasoning process).

(b) Check the result you obtained in part a. Do you find that it is correct?

(c) Find the particular solution of this differential equation that satisfies the condition \( y(0) = 0 \).

**Solution:** By writing the given differential equation as

\[
\int \frac{1}{3 - y} \, dy = \int \, dt
\]

and multiplying both sides by \(-1\) we obtain

\[
\int \frac{1}{y - 3} \, dy = - \int \, dt
\]

and this gives us

\[
\ln |y - 3| = -t + C.
\]

From this we obtain

\[
y - 3 = Ce^{-t}
\]

which gives

\[
y = Ce^{-t} + 3.
\]

This is the general solution. To check that it is correct, let \( y = Ce^{-t} + 3 \). Then

\[
\frac{dy}{dt} = -Ce^{-t}
\]

and

\[
3 - y = 3 - (Ce^{-t} + 3) = -Ce^{-t}
\]

which shows that \( \frac{dy}{dt} = 3 - y \) and hence that the general solution we found is correct.

To find the particular solution for which \( y(0) = 0 \), we solve \( Ce^{0} + 3 = 0 \) to obtain \( C = -3 \). Thus the particular solution is

\[
y = -3e^{-t} + 3.
\]

3. A certain culture of bacteria grows according to

\[
\frac{dP}{dt} = rP
\]

\[
P(0) = P_0
\]

(\text{where, as usual, } P(t) \text{ is the number of cells in the culture at time } t \text{ hours, } P_0 \text{ is the number of cells in the culture at time zero, and } r \text{ is the specific growth rate of the culture}).

It is observed that after 15 minutes there are 510,000 cells in the culture and that after 30 minutes there are 520,200 cells.
(a) Find the initial number of cells \( (P_0) \).
(b) Find the specific growth rate \( (r) \).
(c) Write an expression for \( P(t) \) (the number of cells at any time \( t \geq 0 \)).
(d) How long will it take for the number of cells in the culture to reach one million?

You must show all of your work and write in complete sentences!

**Solution:** We know that the function \( P(t) \) has the form \( P(t) = P_0 e^{rt} \). Since \( P \left( \frac{1}{4} \right) = 510,000 \) and \( P \left( \frac{1}{2} \right) = 520,200 \), we obtain

\[
P_0 e^{\frac{1}{4}r} = 510,000
\]

and

\[
P_0 e^{\frac{1}{2}r} = 520,200.
\]

By division we obtain

\[
\frac{e^{\frac{1}{2}r}}{e^{\frac{1}{4}r}} = \frac{520,200}{510,000}
\]

or

\[
e^{\frac{1}{4}r} = 1.02.
\]

This gives

\[
P_0 = \frac{510,000}{1.02} = 500,000
\]

and

\[
r = 4 \ln (1.02).
\]

Thus \( P(t) = P_0 e^{rt} \) where \( P_0 \) and \( r \) are the numbers given above.

To determine when the population will reach one million, we solve

\[
500,000 e^{rt} = 1,000,000
\]

to obtain

\[
t = \frac{\ln (2)}{r} \approx 8.75.
\]

Thus the population will reach one million after about 8.75 hours.

4. Show, by setting up and evaluating a definite integral, that the area of the shaded region pictured below is 1/3.
Solution: The curves intersect where \( x = 0 \) and where \( x = 1 \). (This is seen by solving the equation \( 1 - x^2 = (x - 1)^2 \).) The area of the shaded region is

\[
\int_0^1 \left( (1 - x^2) - (x - 1)^2 \right) \, dx = \int_0^1 \left( 1 - x^2 - (x^2 - 2x + 1) \right) \, dx
\]

\[
= \int_0^1 (-2x^2 + 2x) \, dx
\]

\[
= \left( \frac{-2}{3}x^3 + x^2 \right) \bigg|_{x=0}^{x=1}
\]

\[
= \frac{-2}{3} + 1
\]

\[
= \frac{1}{3}.
\]

5. The parametric curve

\[
x = t \cos (t) \\
y = t \sin (t) \\
-\pi \leq t \leq \pi
\]

is pictured below.
Show that the length of this curve is given by the definite integral

\[ \int_{-\pi}^{\pi} \sqrt{t^2 + 1} \, dt. \]

(You do not have to evaluate this integral. It is a tough one.)

**Solution:** The arc length is given by

\[ \int_{-\pi}^{\pi} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt. \]

Since

\[ \frac{dx}{dt} = -t \sin(t) + \cos(t) \]

and

\[ \frac{dy}{dt} = t \cos(t) + \sin(t), \]

we see that

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = (-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2 \]

\[ = t^2 \sin^2(t) - 2t \sin(t) \cos(t) + \cos^2(t) \]

\[ + t^2 \cos^2(t) + 2t \sin(t) \cos(t) + \sin^2(t) \]

\[ = t^2 (\sin^2(t) + \cos^2(t)) + \sin^2(t) + \cos^2(t) \]

\[ = t^2 + 1. \]

Thus the arc length is given by the indicated integral. (In case you are curious, the arc length is equal to about 12.22. Try showing this by computing the integral.)