1. Use L’Hospital’s Rule to show that
\[ \lim_{x \to \infty} \frac{\ln(x)}{x} = 0. \]

You must include all details of your reasoning, use correct notation, and write in complete sentences.

**Solution:** First note that \( \lim_{x \to \infty} x = \infty \) which means that we can attempt to use L’Hospital’s Rule. Since
\[ \lim_{x \to \infty} \frac{d}{dx} \left( \ln(x) \right) = \lim_{x \to \infty} \frac{1}{x} = 0, \]
then
\[ \lim_{x \to \infty} \frac{\ln(x)}{x} = 0 \]

by L’Hospital’s Rule.

2. Show that
\[ \int_{3}^{\infty} \frac{\ln(x)}{x} \, dx = \infty. \]

You must include all details of your reasoning (steps in computing integrals, etc.).

**Solution:** To evaluate the indefinite integral
\[ \int \frac{\ln(x)}{x} \, dx, \]
we use the simple substitution \( u = \ln(x) \), \( du = \frac{1}{x} \, dx \). This gives us
\[ \int \frac{\ln(x)}{x} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln(x))^2 + C. \]

Thus we have
\[ \int_{3}^{\infty} \frac{\ln(x)}{x} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{\ln(x)}{x} \, dx \]
\[ = \lim_{t \to \infty} \left( \frac{1}{2} (\ln(t))^2 - \frac{1}{2} (\ln(3))^2 \right) \]
\[ = \infty \]
(because \( \lim_{t \to \infty} \ln(t) = \infty \)).
3. Explain why the series
\[ \sum_{k=3}^{\infty} \frac{\ln (k)}{k} \]
diverges. In doing this, you may refer to the result of problem 2. You must include all details of your reasoning, use correct notation, and write in complete sentences that do not contain the word “it”. (\textit{Hint: Use the Integral Test}.)

\textbf{Solution:} Let \( f \) be the function \( f(x) = \frac{\ln (x)}{x} \) and note that
\[ f'(x) = \frac{1 - \ln (x)}{x^2} \]
from which we see that \( f'(x) < 0 \) throughout the interval \((e, \infty)\). This means that \( f \) is decreasing on the interval \([3, \infty)\). (Note that \( f \) is also positive and continuous on this interval.) In problem 2, we showed that the improper integral
\[ \int_{3}^{\infty} \frac{\ln (x)}{x} \, dx \]
diverges. Thus the series
\[ \sum_{k=3}^{\infty} \frac{\ln (k)}{k} \]
diverges by the Integral Test.

4. Suppose that \( a_n \) and \( b_n \) are sequences such that \( 0 \leq a_n \leq b_n \) for all \( n \geq 1 \). Decide whether each of the following statements \textbf{must be true} or \textbf{is not necessarily true}. (Circle the correct choice.)

(a) If \( \lim_{n \to \infty} a_n = 1 \), then \( \sum_{n=1}^{\infty} b_n \) diverges.
   \( \square \) \text{must be true} / \text{is not necessarily true}.
(b) If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) also converges.
   \( \square \) \text{must be true} / \text{is not necessarily true}.
(c) If \( \lim_{n \to \infty} b_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge.
   \( \square \) \text{must be true} / \text{is not necessarily true}.
(d) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \), then both \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) must be divergent.
   \( \square \) \text{must be true} / \text{is not necessarily true}.
(e) If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) also diverges.
   \( \square \) \text{must be true} / \text{is not necessarily true}.

5. Use the Alternating Series Test to explain why the series
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}} \]
is convergent. (Write in complete sentences. Do not use the word “it”. In your explanation, make sure to verify that all of the hypotheses of the Alternating Series Test are satisfied.)

**Explanation:** First note that the given series is an alternating series and also note that the sequence

\[ (-1)^{n+1} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} \]

is monotone decreasing to the limit 0. (This is because the sequence \( \sqrt{n} \) is monotone increasing to \( \infty \).) Therefore, by the Alternating Series Test, the series

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}} \]

converges.

6. Use the Ratio Test to explain why the series

\[ \sum_{n=1}^{\infty} \frac{n!}{8^n} \]

diverges. (Write in complete sentences. Do not use the word “it”.)

**Explanation:** Note that this series has all positive terms. Also

\[ \frac{a_{n+1}}{a_n} = \frac{(n + 1)!}{8^{n+1}} \cdot \frac{8^n}{n!} = \frac{n + 1}{8} \]

Since

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n + 1}{8} = \infty, \]

the Ratio Test tells us that the series

\[ \sum_{n=1}^{\infty} \frac{n!}{8^n} \]

diverges.