1. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n.$$

You must include all details of your work. Write in complete sentences that do not contain the word “it”.

Write your conclusions here:
The radius of convergence of this power series is ___________.
The interval of convergence of this power series is ___________.

Show your work here:
Solution: The absolute value of the ratio of the \((n+1)\)st term to the \(n\)th term of this series is

$$\left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| = \frac{n+1}{n+2} \cdot |x|.$$ 

Since

$$\lim_{n \to \infty} \frac{n+1}{n+2} = 1,$$

we see that

$$\lim_{n \to \infty} \frac{n+1}{n+2} \cdot |x| = |x|.$$ 

Thus the given power series converges for all real numbers \(x\) such that \(|x| < 1\). This means that the radius of convergence of the given series is 1 and that the series converges for all \(x\) in the interval \((-1, 1)\).

We now test the series at the endpoints of the interval of convergence.

At \(x = -1\), we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (-1)^n = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$ 

This series can be seen to diverge by comparison with the divergent series \(\sum_{n=2}^{\infty} \frac{1}{n}\) (by the Limit Comparison Test).
At $x = 1$, we have the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n,$$

This series can be seen to converge by the Alternating Series Test.

In conclusion, the interval of convergence of the given power series is $(-1, 1]$.

2. (a) Begin with the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

for all $x$ such that $|x| < 1$ and make a substitution (or substitutions) to obtain

the power series representation of

$$\frac{1}{4x + 1}.$$

(b) Determine the radius of convergence of the power series your found in part a.

Solution:

$$\frac{1}{4x + 1} = \frac{1}{1 - (-4x)}$$

$$= \sum_{n=0}^{\infty} (-4x)^n$$

$$= 1 - 4x + 16x^2 - 64x^3 + 256x^4 - \cdots$$

for all $x$ such that $|4x| < 1$ (or, equivalently, all $x$ such that $|x| < \frac{1}{4}$). Thus the radius of convergence of the given series is $1/4$.

3. Find the Maclaurin Series of the function

$$f(x) = xe^x$$

and also find the radius of convergence of this series.

Solution: Note that

$$f^{(0)}(x) = xe^x \quad \text{and} \quad f^{(0)}(0) = 0$$

$$f^{(1)}(x) = xe^x + e^x = (x + 1)e^x \quad \text{and} \quad f^{(1)}(0) = 1$$

$$f^{(2)}(x) = (x + 2)e^x \quad \text{and} \quad f^{(2)}(0) = 2.$$
Since 
\[
\frac{x^{n+1}}{n!} \cdot \frac{(n-1)!}{x^n} = \frac{1}{n} |x|
\]
which approaches 0 as \( n \to \infty \) (no matter what the value of \( x \)) and since \( 0 < 1 \), then the Ratio Test tells us that this power series converges for all real numbers \( x \). Thus the radius of convergence is \( \infty \).

4. Suppose that
\[
\sum_{n=0}^{\infty} a_n (x - 3)^n = a_0 + a_1 (x - 3) + a_2 (x - 3)^2 + a_3 (x - 3)^3 + \cdots
\]
is a power series and suppose that we know that this power series converges when \( x = -4 \). Decide whether each of the following statements must be true or is not necessarily true. (Circle the correct choice.)

(a) This power series converges when \( x = -5 \).
   (must be true / is not necessarily true).

(b) This power series converges when \( x = 10 \).
   (must be true / is not necessarily true).

(c) This power series converges when \( x = -4.5 \).
   (must be true / is not necessarily true).

(d) This power series converges when \( x = 9 \).
   (must be true / is not necessarily true).

(e) This power series converges when \( x = 10.5 \).
   (must be true / is not necessarily true).

The grading of this question will be as follows:

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5. (a) Compute the following generalized binomial coefficients.

1. \( \binom{-3}{0} = 1 \)

2. \( \binom{-3}{1} = -3 \)

3. \( \binom{-3}{2} = \frac{(-3)(-4)}{2!} = 6 \)

4. \( \binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10 \)

5. \( \binom{-3}{4} = \frac{(-3)(-4)(-5)(-6)}{4!} = 15 \)
(b) Use the computations that you did in part a to write the first five terms of the MacClaurin Series (binomial expansion) of \((1 + x)^{-3}\). (You don’t have to write out the whole series – just the first five terms).

**Answer:** The beginning of the MacClaurin Series for \((1 + x)^{-3}\) is

\[1 - 3x + 6x^2 - 10x^3 + 15x^4.\]