When the region pictured in Figure A is revolved about the $y$ axis, we obtain the solid pictured in Figure B (a bowl with a hole in the bottom).

a) Use the washer method to obtain a definite integral that gives the volume of this solid.
b) Use the shell method to obtain a definite integral that gives the volume of this solid.

c) Evaluate one of the integrals that you obtained in order to find the volume of the solid.

Solution:

a) Washer Method

\[
V = \pi \int_{0}^{1} (x_2^2 - x_1^2) \, dy
\]

\[
= \pi \int_{0}^{1} \left( (y^{1/4} + 1)^2 - (y^{1/2} + 1)^2 \right) \, dy
\]

\[
= \pi \int_{0}^{1} \left( y^{1/2} + 2y^{1/4} + 1 - (y + 2y^{1/2} + 1) \right) \, dy
\]

\[
= \pi \int_{0}^{1} \left( -y^{1/2} + 2y^{1/4} - y \right) \, dy
\]

\[
= \pi \left( -\frac{2}{3}y^{3/2} + \frac{8}{5}y^{5/4} - \frac{1}{2}y^2 \right)\bigg|_{y=1}^{y=0}
\]

\[
= \pi \left( -\frac{20}{30} + \frac{48}{30} - \frac{15}{30} \right)
\]

\[
= \frac{13\pi}{30}.
\]
b) Shell Method

\[ V = 2\pi \int_{1}^{2} x (y_u - y_l) \, dx \]

\[ = 2\pi \int_{1}^{2} x ((x - 1)^2 - (x - 1)^4) \, dx \]

\[ = 2\pi \int_{0}^{1} (u + 1) (u^2 - u^4) \, du \]

\[ = 2\pi \int_{0}^{1} (u^3 - u^5 + u^2 - u^4) \, du \]

\[ = 2\pi \left( \frac{1}{4} u^4 - \frac{1}{6} u^6 + \frac{1}{3} u^3 - \frac{1}{5} u^5 \right) \bigg|_{u=1}^{u=0} \]

\[ = 2\pi \left( \frac{15}{60} - \frac{10}{60} + \frac{20}{60} - \frac{12}{60} \right) \]

\[ = \frac{13\pi}{30}. \]