Series with Both Positive and Negative Terms

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1 Absolute Convergence

Consider the series

$$
\sum_{k=1}^{\infty} a_k = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} - \cdots.
$$

The general ($k$th) term in this series has absolute value $1/k^2$ (that is, $|a_k| = 1/k^2$) and the series consists of three positive terms followed by three negative terms followed by three positive terms and so on (continuing forever in this fashion).

Does the above series converge? To answer this question, we need to consider three other series:

$$
\sum_{k=1}^{\infty} |a_k| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots,
$$

$$
\sum_{k=1}^{\infty} 2|a_k| = 2 \cdot \frac{1}{1^2} + 2 \cdot \frac{1}{2^2} + 2 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{4^2} + 2 \cdot \frac{1}{5^2} + 2 \cdot \frac{1}{6^2} + \cdots,
$$

and

$$
\sum_{k=1}^{\infty} (|a_k| - a_k) = 0 + 0 + 0 + 2 \cdot \frac{1}{4^2} + 2 \cdot \frac{1}{5^2} + 2 \cdot \frac{1}{6^2} + \cdots.
$$
The first of the above three series is the $p$–series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We are already familiar with this series and we know that it converges (having proved this using the Integral Test).

The second of the above series is $\sum_{k=1}^{\infty} 2 \cdot \frac{1}{k^2}$. Since each term of this series is a constant multiple of the corresponding term of the $p$–series, this series also converges.

Finally, each term in the third series is non–negative and is less than or equal to the corresponding term in the second series. This means that this third series also converges by the Standard Comparison Test. Hence, all three of the above series converge.

To prove that the original series, $\sum_{k=1}^{\infty} a_k$, also converges, we now simply note that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (|a_k| - (|a_k| - a_k)).$$

In other words, each term of the series $\sum_{k=1}^{\infty} a_k$ is the difference of corresponding terms in the series $\sum_{k=1}^{\infty} |a_k|$ and $\sum_{k=1}^{\infty} (|a_k| - a_k)$. However, both of the latter series are convergent. This means that the series $\sum_{k=1}^{\infty} a_k$ is also convergent. (Recall that if a series $\sum c_k$ and a series $\sum d_k$ are both convergent, then the the series $\sum (c_k - d_k)$ must also be convergent.)

The reasoning that has been used in the above example can be generalized: Suppose that we have a series, $\sum a_k$, that might have both positive and negative terms, and suppose also that we know that the series $\sum |a_k|$ is convergent. Then we also know that the series $\sum 2 |a_k|$ is convergent. Furthermore, the series $\sum (|a_k| - a_k)$ has all non–negative terms and

$$0 \leq |a_k| - a_k \leq 2 |a_k|$$

for all $k$. (In fact $|a_k| - a_k$ must always be equal to 0 or to $2 |a_k|$.) Since the series $\sum 2 |a_k|$ is convergent, then the series $\sum (|a_k| - a_k)$ is also convergent by the Standard Comparison Test. Upon noting that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (|a_k| - (|a_k| - a_k)),$$

we see that the series $\sum_{k=1}^{\infty} a_k$ must also be convergent. In summary, if the series $\sum_{k=1}^{\infty} |a_k|$ is convergent, then the series $\sum_{k=1}^{\infty} a_k$ must also be convergent. Since this fact is very useful in the study of series, we will state it as a theorem.
Theorem 1  If the series \( \sum |a_k| \) is convergent, then the series \( \sum a_k \) is also convergent.

Let us look at another example in which we can use Theorem 1.

Example 2  Explain why the series

\[
\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} - \frac{1}{2^9} - \frac{1}{2^{10}} + \cdots
\]

converges.

Answer: If we call the above series \( \sum_{k=1}^{\infty} a_k \), then

\[
\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k
\]

and we know the latter series is a convergent geometric series (with \( r = 1/2 \)). We can then conclude from Theorem 1 that the series \( \sum_{k=1}^{\infty} a_k \) is also convergent. In fact, since we know that

\[
\sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k = 1,
\]

we can conclude that

\[
\sum_{k=1}^{\infty} a_k < 1.
\]

(It might make a fun exercise to try to compute the exact value of \( \sum_{k=1}^{\infty} a_k \).)

It is important to note that Theorem 1 tells us that if \( \sum |a_k| \) converges, then it must be true that \( \sum a_k \) also converges. However, if it turns out that the series \( \sum_{k=1}^{\infty} |a_k| \) diverges, then Theorem 1 does not tell us anything about the series \( \sum_{k=1}^{\infty} a_k \). It is possible that the series \( \sum_{k=1}^{\infty} |a_k| \) is divergent but the series \( \sum_{k=1}^{\infty} a_k \) is convergent! For example, in the next section we will prove that the series

\[
\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots
\]
(which is known as the alternating harmonic series) is convergent. This is true even though we know that the harmonic series,
\[ \sum_{k=1}^{\infty} |a_k| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \]
is divergent.

We conclude this section by introducing some vocabulary: If we have a series, \( \sum a_k \), for which the series \( \sum |a_k| \) is convergent, then we say that the series \( \sum a_k \) is \textbf{absolutely convergent} or \textbf{convergent in absolute value}. Theorem 1 tells us that any absolutely convergent series must be convergent. If, however, we have a series, \( \sum a_k \), which is convergent but for which the series \( \sum |a_k| \) is divergent, then we say that the series \( \sum a_k \) is \textbf{conditionally convergent}. Thus, for example, the series
\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} - \cdots \]
and
\[ \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} - \frac{1}{2^9} - \frac{1}{2^{10}} + \cdots \]
are both absolutely convergent, but the series
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \]
is not absolutely convergent. (In the upcoming section, we will prove that the latter series is convergent, meaning that this series is conditionally convergent.)

2 \ The Alternating Series Test

We begin this section by proving that the alternating harmonic series,
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots, \]
converges. We then generalize our argument to obtain a theorem that is known as the Alternating Series Test.
First let us consider the alternating harmonic series (given above). This series is not absolutely convergent, so we cannot apply Theorem 1 to it. Instead, we will examine its sequence of partial sums: Since

\[ s_1 = 1 \]
\[ s_2 = 1 - \frac{1}{2}, \]

we easily observe that

\[ s_2 < s_1. \]

Also, since

\[ s_3 = 1 - \frac{1}{2} + \frac{1}{3} \]

we see that

\[ s_3 = \left(1 - \frac{1}{2}\right) + \frac{1}{3} = s_2 + \frac{1}{3} > s_2 \]

and

\[ s_3 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) = s_1 - (\text{something positive}) < s_1 \]

from which we conclude that

\[ s_2 < s_3 < s_1. \]

Now note that

\[ s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \]

so

\[ s_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = s_2 + (\text{something positive}) > s_2 \]

and

\[ s_4 = \left(1 - \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4} = s_3 - (\text{something positive}) < s_3. \]

We now have that

\[ s_2 < s_4 < s_3 < s_1. \]

Just to make sure we understand how this sequence of partial sums is ordered, let’s consider \( s_5 \):

\[ s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \]
gives us
\[ s_5 = \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right) + \frac{1}{5} = s_4 + \text{(something positive)} > s_4 \]
and
\[ s_5 = \left( 1 - \frac{1}{2} + \frac{1}{3} \right) - \left( \frac{1}{4} - \frac{1}{5} \right) = s_3 - \text{(something positive)} < s_3. \]

We now have
\[ s_2 < s_4 < s_5 < s_3 < s_1. \]

By continuing with this sort of reasoning, we observe that
\[ s_2 < s_4 < s_6 < s_8 < \cdots < s_7 < s_5 < s_3 < s_1. \]

Note that the sequence of partial sums is bounded. (A lower bound for the sequence is \( s_2 \) and an upper bound is \( s_1 \).) Also, the sequence of even–numbered partial sums is monotone increasing and the sequence of odd–numbered partial sums is monotone decreasing. Thus, both of these sequences have limits. In addition, if we look at any two consecutive partial sums,
\[ s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \frac{1}{n} \]
and
\[ s_{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \frac{1}{n} + (-1)^{n+2} \frac{1}{n+1}, \]
we see that
\[ |s_{n+1} - s_n| = \left| (-1)^{n+2} \frac{1}{n+1} \right| = \frac{1}{n+1} \]
which shows us that
\[ \lim_{n \to \infty} |s_{n+1} - s_n| = \lim_{n \to \infty} \frac{1}{n+1} = 0. \]

This tells us that the distance between the even–numbered and odd–numbered partial sums approaches zero as \( n \to \infty \). This, in turn, tells us that the limit of each of these sequences (the even–numbered and odd–numbered partial sums) must be the same! In other words the entire sequence of partial sums converges to some definite limit. Therefore, the alternating harmonic series
converges. Perhaps some numerical computations are in order here to help see what is happening in the above reasoning:

\[
s_1 = 1
\]

\[
s_2 = 1 - \frac{1}{2} = 0.5
\]

\[
s_3 = 1 - \frac{1}{2} + \frac{1}{3} = 0.833
\]

\[
s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 0.583
\]

\[
s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.783
\]

\[
s_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 0.616
\]

\[
s_7 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} = 0.759523809.
\]

The reasoning that was used in showing that the alternating harmonic series converges can be generalized. We will state this result as Theorem 3. First, let us define exactly what we mean by an alternating series. An alternating series is a series in which every other term is positive and every other term is negative. Thus an alternating series is a series,

\[
a_1 + a_2 + a_3 + \cdots,
\]

for which either all of the odd–numbered terms are positive and all of the even–numbered terms are negative or vice–versa.

**Theorem 3 (Alternating Series Test)** Suppose that \( \sum {a_k} \) is an alternating series and suppose that the sequence \( |a_n| \) is monotone decreasing and that \( \lim_{n \to \infty} |a_n| = 0 \). Then the series \( \sum {a_k} \) converges.

**Proof.** Let us assume that all of the odd–numbered terms of the series \( \sum_{k=1}^{\infty} a_k \) are positive numbers and that all of the even–numbered terms are negative numbers. (The proof would be entirely similar if it were the other way around.) Also, let \( s_n \) be the sequence of partial sums of \( \sum_{k=1}^{\infty} a_k \).

Observe that

\[
s_1 = a_1
\]

\[
s_2 = a_1 + a_2 = s_1 + (\text{something negative}) < s_1
\]
s_2 < s_1.

Next, observe that
\[ s_3 = a_1 + a_2 + a_3 \]
so
\[ s_3 = (a_1 + a_2) + |a_3| = s_2 + \text{(something positive)} > s_2 \]
and
\[ s_3 = a_1 + (-|a_2| + |a_3|) = s_1 + \text{(something negative)} < s_1. \]
(Note the “something negative” in the above calculation is \(|a_3| - |a_2|\). We know that this number is negative because we are assuming that the sequence \(|a_n|\) is monotone decreasing, meaning that \(|a_3| < |a_2|\).) We now have
\[ s_2 < s_3 < s_1. \]

If we continue with this type of reasoning, we see that
\[ s_2 < s_4 < s_6 < s_8 < \cdots < s_7 < s_5 < s_3 < s_1. \]

This shows that the sequence \(s_n\) is bounded and it also shows that the sequence of even-numbered partial sums is monotone increasing (and hence has a limit) and the sequence of odd-numbered partial sums is monotone decreasing (and hence has a limit). The fact that the limits of these two subsequences must be the same follows from the fact that
\[ \lim_{n \to \infty} |s_{n+1} - s_n| = \lim_{n \to \infty} |a_{n+1}| = 0. \]

This completes the proof of the theorem. ■

We now provide another example to illustrate the Alternating Series Test.

**Example 4** Let us use show that the series
\[ \sum_{k=3}^{\infty} (-1)^{k+1} \frac{\ln (k)}{k} = \frac{\ln (3)}{3} - \frac{\ln (4)}{4} + \frac{\ln (5)}{5} - \frac{\ln (6)}{6} + \cdots \]
converges.

First, we observe that this series is indeed an alternating series: The numbers \(\ln (k)/k\) are all positive numbers and the \((-1)^{k+1}\) makes the series alternate.
Now consider the sequence

\[ (-1)^{n+1} \frac{\ln(n)}{n} = \frac{\ln(n)}{n}. \]

If we let \( f(x) = \frac{\ln(x)}{x} \), then

\[ f'(x) = \frac{1 - \ln(x)}{x^2} < 0 \quad \text{for all } x \geq e. \]

This shows that the function \( f \) is monotone decreasing on the interval \([e, \infty)\) and hence that the sequence \( \frac{\ln(n)}{n} \) is monotone decreasing for all \( n \geq 3 \).

Finally, note that

\[ \lim_{n \to \infty} \frac{\ln(n)}{n} = \lim_{x \to \infty} \frac{\ln(x)}{x} = 0 \quad \text{(by L'Hopital's Rule)}. \]

Since all of the hypothesis of Theorem 3 are satisfied, we conclude that the series

\[ \sum_{k=3}^{\infty} (-1)^{k+1} \frac{\ln(k)}{k} \]

converges.

**Remark 5** The Alternating Series Test can **never** be used to conclude that a series diverges!

### 3 The Ratio Test

Let us recall the basic facts about a geometric series

\[ \sum r^k \]

where \( r \) is a constant and the summation index is \( k \). We know that the geometric series converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \). Furthermore, in the case of convergence, we can actually compute the sum of the series. This sum depends on where (which value of \( k \)) we begin adding. For example,

\[ \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \]
and
\[ \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}. \]

An important observation about a geometric series is that the ratios of successive terms of the series are constant. This is easy to see. For the geometric series
\[ \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots, \]
the \( n \text{th} \) term in the series is \( a_n = r^n \) and the \((n+1)\text{st}\) term in the series is \( a_{n+1} = r^{n+1} \). Therefore the ratio of the \((n+1)\text{st}\) term to the \( n \text{th} \) term is
\[ \frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r \]
which is constant.

The **Ratio Test** is a tool for studying series that are “like” geometric series. By “like” geometric series, we mean that the ratio of the absolute value of the \((n+1)\text{st}\) term to the \( n \text{th} \) term of the series is “almost constant” for all large values of \( n \). To be more specific, suppose that we have a series
\[ \sum_{k=1}^{\infty} a_k \]
(none of whose terms is zero) and suppose that we know that
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \]
where \( r \) is a real number. This means that
\[ \left| \frac{a_{n+1}}{a_n} \right| \approx r \]
for all large values of \( n \). Now consider the sequence of partial sums of the
series $\sum_{k=1}^{\infty} |a_k|$: 

$s_1 = |a_1| = |a_1| (1)$

$s_2 = |a_1| + |a_2| = |a_1| + |a_1| \left( \frac{|a_2|}{|a_1|} \right) = |a_1| \left( 1 + \frac{|a_2|}{|a_1|} \right)$

$s_3 = |a_1| + |a_2| + |a_3| = |a_1| + |a_1| \left( \frac{|a_2|}{|a_1|} \right) + |a_1| \left( \frac{|a_3|}{|a_2|} \right) = |a_1| \left( 1 + \frac{|a_2|}{|a_1|} + \frac{|a_2|}{|a_1|} \frac{|a_3|}{|a_2|} \right)$

$s_4 = |a_1| \left( 1 + \frac{|a_2|}{|a_1|} + \frac{|a_2|}{|a_1|} \frac{|a_3|}{|a_2|} + \frac{|a_2|}{|a_1|} \frac{|a_3|}{|a_2|} \frac{|a_4|}{|a_3|} \right)$

etc.

If every one of the ratios $|a_{n+1}|/|a_n|$ is approximately equal to the constant $r$, then, from the above calculation of partial sums, we see that

$s_1 = |a_1| (1)$

$s_2 \approx |a_1| (1 + r)$

$s_3 \approx |a_1| (1 + r + r^2)$

$s_4 \approx |a_1| (1 + r + r^2 + r^3)$

etc.

Thus, the series $\sum_{k=1}^{\infty} |a_k|$ “looks like” the geometric series $\sum_{k=1}^{\infty} |a_1| r^k$. Of course, we are only assuming that

$$\left| \frac{a_{n+1}}{a_n} \right| \approx r$$

for all large values of $n$, but that does not affect our argument about convergence/divergence of $\sum |a_k|$, since convergence versus divergence of a series does not depend on the first finitely many terms of the series.

The precise statement of the Ratio Test is given in the following theorem.

**Theorem 6 (Ratio Test)** Let $\sum a_k$ be a series, none of whose terms is 0, and suppose that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r.$$
1. If $r < 1$, then the series $\sum |a_k|$ converges (and hence $\sum a_k$ also converges).

2. If $r > 1$ (including the possibility that $r = \infty$), then $\sum a_k$ diverges.

Remark 7 If 
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,
\]
then the Ratio Test tells us nothing about convergence/divergence of $\sum a_k$.

We conclude with three examples that illustrate the Ratio Test.

Example 8 In this example, we will show that the series
\[
\sum_{k=1}^{\infty} \left( \frac{k + 2}{2k} \right)^k = \frac{3}{2} + 1 + \left( \frac{5}{6} \right)^3 + \left( \frac{6}{8} \right)^4 + \cdots
\]
converges.

The key here is to consider the ratios of successive terms in the series. We will begin by doing some specific computations. (Note that since all terms of this series are positive, then $|a_{n+1}/a_n| = a_{n+1}/a_n$ for all $n$.)

\[
\begin{align*}
a_2 &= \frac{1}{\frac{3}{2}} = \frac{2}{3} = 0.6 \\
a_3 &= \frac{\left( \frac{5}{6} \right)^3}{1} = \left( \frac{5}{6} \right)^3 \approx 0.578 \\
a_4 &= \frac{\left( \frac{6}{8} \right)^4}{\left( \frac{5}{6} \right)^3} = \frac{6}{8} \cdot \left( \frac{5}{6} \right)^3 = \frac{6}{8} \cdot \left( \frac{36}{40} \right)^4 \approx 0.492 \\
a_5 &= \frac{\left( \frac{7}{10} \right)^5}{\left( \frac{6}{8} \right)^4} = \frac{7}{10} \cdot \left( \frac{6}{8} \right)^4 = \frac{7}{10} \cdot \left( \frac{56}{60} \right)^4 \approx 0.531 \\
a_6 &= \frac{\left( \frac{8}{12} \right)^6}{\left( \frac{7}{10} \right)^5} = \frac{8}{12} \cdot \left( \frac{80}{84} \right)^5 \approx 0.522 \\
a_7 &= \frac{\left( \frac{9}{14} \right)^7}{\left( \frac{8}{12} \right)^6} = \frac{9}{14} \cdot \left( \frac{108}{112} \right)^6 \approx 0.517 \\
a_8 &= \frac{\left( \frac{10}{16} \right)^8}{\left( \frac{9}{14} \right)^7} = \frac{10}{16} \cdot \left( \frac{140}{144} \right)^7 \approx 0.513
\end{align*}
\]
By looking at the above computations, it appears that perhaps
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}
\]
(and we will show that this is in fact true).

Since the $n$th term in the series is
\[
a_n = \left(\frac{n + 2}{2n}\right)^n
\]
and the $(n + 1)$st term in the series is
\[
a_{n+1} = \left(\frac{(n + 1) + 2}{2(n + 1)}\right)^{n+1} = \left(\frac{n + 3}{2n + 2}\right)^{n+1},
\]
we see that
\[
\frac{a_{n+1}}{a_n} = \left(\frac{n + 3}{2n + 2}\right) \left(\frac{n + 3}{2n + 2}\right)^n = \frac{n + 3}{2n + 2} \cdot \left(\frac{2n + 3}{n + 2}\right)^n = \frac{n + 3}{2n + 2} \cdot \left(\frac{2n^2 + 6n}{2n^2 + 6n + 4}\right)^n
\]

Our goal is to show that \(\lim_{n \to \infty} |a_{n+1}/a_n| = 1/2\). First, note that
\[
\lim_{n \to \infty} \frac{n + 3}{2n + 2} = \lim_{n \to \infty} \frac{1 + \frac{3}{n}}{2 + \frac{2}{n}} = \frac{1}{2}.
\]

Thus, what we need to do is study the limit problem
\[
\lim_{n \to \infty} \left(\frac{2n^2 + 6n}{2n^2 + 6n + 4}\right)^n
\]
(which is a \(1^\infty\) indeterminate form problem) and show that this limit is equal to 1. This will then give us that
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n + 3}{2n + 2} \cdot \left(\frac{2n^2 + 6n}{2n^2 + 6n + 4}\right)^n = \frac{1}{2} \cdot 1 = \frac{1}{2}.
\]
In order to use L’Hopital’s Rule, let

\[ y = \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right)^x. \]

Then

\[ \ln (y) = x \ln \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right) \]

so

\[ \ln (y) = \frac{\ln \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right)}{\frac{1}{x}}. \]

Since

\[ \lim_{x \to 1} \ln \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right) \]

is a 0/0 indeterminate form problem, we may try to apply L’Hopital’s Rule. Consider

\[
\begin{align*}
\lim_{x \to \infty} & \frac{2x^2+6x+4}{2x^2+6x} \cdot \frac{(2x^2+6x+4)(4x+6)-(2x^2+6x)(4x+6)}{(2x^2+6x+4)^2} \\
&= \lim_{x \to \infty} \frac{1}{\frac{2x^2+6x}{2x^2+6x+4}} \cdot \frac{4(4x+6)}{-\frac{1}{x^2}} \\
&= \lim_{x \to \infty} \frac{4(4x+6)}{4x^4 + 24x^3 + 44x^2 + 24x} \\
&= 0.
\end{align*}
\]

This tells us that

\[ \lim_{x \to \infty} \ln \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right) = 0 \]

(by L’Hopital’s Rule) and hence that

\[ \lim_{x \to \infty} \ln (y) = 0. \]

Therefore

\[ \lim_{x \to \infty} y = \lim_{x \to \infty} \left( \frac{2x^2 + 6x}{2x^2 + 6x + 4} \right)^x = 1. \]

and it then follows that

\[ \lim_{n \to \infty} \left( \frac{2n^2 + 6n}{2n^2 + 6n + 4} \right)^n = 1. \]
We have now proved that
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}.
\]

Since \(1/2 < 1\), the series
\[
\sum_{k=1}^{\infty} \left( \frac{k+2}{2k} \right)^k
\]
converges by the Ratio Test.

**Example 9** We will use the Ratio Test to show that the series
\[
\sum_{k=1}^{\infty} \frac{k^k}{k!} = 1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \frac{4^4}{4!} + \cdots
\]
diverges. (Actually, the easiest way to show that this series diverges is just to use the Basic Divergence Test, but we will use the Ratio Test.)

Here we have
\[
a_n = \frac{n^n}{n!}
\]
and
\[
a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}
\]
so
\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}
\]
\[
= \frac{(n+1)^n}{n!} \cdot \frac{n!}{n^n}
\]
\[
= \frac{(n+1)^n}{n^n}
\]
\[
= \left(1 + \frac{1}{n} \right)^n.
\]

The limit problem
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n
\]
is of the $1^\infty$ indeterminate form. We can however use L’Hopital’s Rule to do this problem and we find that the answer is $e$, which is greater than 1. Therefore the series

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

diverges by the Ratio Test.

**Example 10 (an example where the Ratio Test does not apply)** If we consider either the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

or the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k},$$

then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1/n+1}{1/n} = \frac{n}{n+1}$$

and we see that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n}{n+1} = 1.$$ 

Since this limit is equal to 1, we cannot use the Ratio Test to draw any conclusions about either the harmonic series or the alternating harmonic series. As we have shown by other means, the harmonic series diverges and the alternating harmonic series converges. This example serves to illustrate the point that the Ratio Test gives no information about convergence/divergence if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$