Using Integrals to Compute Volumes

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General Principle

\[ V = \int_a^b A(y) \, dy \]

\[ A(y) = \text{area of slice} \]
Example 1

Find the volume of a sphere of radius R.
Example 2

Find the volume of a pyramid with square base of side length \(b\) and height \(h\).
Volumes of Solids of Revolution

We will consider two methods for computing volumes of solids of revolution.
• The Disk (or Slab) Method
• The Shell (or Cylinder) Method

Depending on the problem, one of these methods may be easier to use than the other. (In some problems, both methods are about equally good.)
The Disk (or Slab) Method

Suppose that a plane region (pictured in red) is revolved about the x axis.

The volume of the solid of revolution is

\[ V = \pi \int_a^b (y_o^2 - y_i^2) \, dx = \pi \int_a^b \left( (f(x))^2 - (g(x))^2 \right) \, dx \]
Example 3

Find the volume of the solid of revolution when the curve region bounded by the curve $y = x^2$ and the line $x = 1$ is revolved about the $x$ axis. Also find the volume of the solid of revolution when this same curve is revolved about the $y$ axis.
Example 4

Find the volume of a “cap” of height \( h \) at the top of a sphere of radius \( R \).
The Shell (or Cylinder) Method

Suppose that a plane region (pictured in red) is revolved about the y axis.

The volume of the solid of revolution is

\[ V = 2\pi \int_a^b x(y_u - y_l) \, dx = 2\pi \int_a^b x(f(x) - g(x)) \, dx \]
Example 5

Repeat Example 3 using the shell method:

\[ y = x^2 \]

\( x = 1 \)
Example 6

A hole of radius $r$ is drilled through the center of a sphere of radius $R$ (where $r<R$). Find the volume that remains.