Line Integrals
Line Integral of a Function

Let $C$ be a smooth curve.
Let $\mathbf{r}(t)$, $a \leq t \leq b$, be a parameterization of $C$.
Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function that is continuous on some domain containing the curve $C$.
The line integral of $f$ along $C$ (with respect to arc length) is computed as follows:

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| \, dt$$
If $f(x, y) \geq 0$ for all $(x, y) \in C$, the line integral $\int_C f(x, y) \, ds$ gives us the area of the “sheet” or “curtain” that is bounded below by the curve $C$ and bounded above by the curve $z = f(x(t), y(t))$. 
Example: Use a line integral to find the surface area of the cylinder $x^2 + y^2 = 1, \ 0 \leq z \leq 1$. 

The Cylinder $x^2 + y^2 = 1, \ 0 \leq z \leq 1$
Example: Use a line integral to find the surface area of the portion of the cylinder $x^2 + y^2 = 1$ that lies above the plane $z = 0$ and lies below the plane $x + 2z = 1$.

The surface (plane) $z = \frac{1}{2} (1 - x)$

Cylinder $x^2 + y^2 = 1$
Line integrals can be used, not just to find areas, but also to other things such as find masses and centers of mass of wires. In particular, if $C$ is thought of as a wire with density function $\rho$, then the mass of the wire is $\int_C \rho(x,y) \, ds$ and the center of mass of the wire is at $(\overline{x}, \overline{y})$ where $\overline{x} = \int_C x \rho(x,y) \, ds$ and $\overline{y} = \int_C y \rho(x,y) \, ds$. 
Example

Find the mass and the center of mass of the semicircular piece of wire that consists of the quarter of the unit circle lying in the first quadrant with density function $\rho(x,y) = x$. 
Line Integrals of Vector Fields

If $C$ is a curve parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$, and $\mathbf{F}$ is a vector field, then the line integral of $\mathbf{F}$ along $C$ is defined to be the line integral of the component of $\mathbf{F}$ in the direction of the unit tangent vector to $C$. 
The vector $\mathbf{F}(x, y)$ is the force vector along the curve $C$. The unit tangent vector $\mathbf{T}$ to $C$ at any point $(x, y)$ is given by $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$. The component of $\mathbf{F}$ in the direction of $\mathbf{T}$ is $\text{comp}_T \mathbf{F} = \frac{\mathbf{F} \cdot \mathbf{T}}{|\mathbf{T}|} = \mathbf{F} \cdot \mathbf{T}$.

Also, $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ so

$$
\int_C \text{comp}_T \mathbf{F} \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
$$
Let $\mathbf{F}$ be the vector field picture below.
Let $C_1$ be the unit circle traversed counterclockwise.
Let $C_2$ be the unit circle traversed clockwise.
Let $C_3$ be the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$.
Explain why:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} < 0$$
$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} > 0$$
$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$$
The vector field given in the previous example is

\[ F(x,y) = yi - xj. \]

Find the exact values of each of the integrals:
Example: Work

- Scenario 1: Man pushes crate along floor and then takes elevator.
- Scenario 2: Man pushes crate up ramp.
- Find the work done by the gravity vector field in each scenario.