Vectors
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A vector is an equivalence class of directed line segments – all having the same length and pointing in the same direction.
Consider the directed line segment $\overrightarrow{AB}$, joining the point $A(-3, 4)$ to the point $B(1, -2)$. and the directed line segment $\overrightarrow{CD}$, joining the point $C(1, 5)$ to the point $D(5, -1)$.

The directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length and point in the same direction. Each of these directed line segments is a representative of the same vector.

Any directed line segment having the same length and pointing in the same direction as $\overrightarrow{AB}$ is also a representative of this same vector. Any vector has infinitely many representatives.
For any vector, \( \mathbf{a} \), there is a representative of \( \mathbf{a} \) that is based at the origin, \( O \). This representative of \( \mathbf{a} \) is called the **standard representative** of \( \mathbf{a} \).

For example, the standard representative of the vector studied in the previous example is the directed line segment \( \overrightarrow{OE} \) where \( O \) is the point \( O(0, 0) \) and \( E \) is the point \( E(4, -6) \). We can write the vector \( \mathbf{a} \) as \( \mathbf{a} = \langle 4, -6 \rangle \).
The Sum of Two Vectors

\[ \mathbf{a} + \mathbf{b} \]
Scalar Multiple of a Vector

- \( a \)
- \( 2a \)
- \( -1.5a \)
If \( \mathbf{a} = \langle x_1, y_1, z_1 \rangle \) and \( \mathbf{b} = \langle x_2, y_2, z_2 \rangle \), then
\[
\mathbf{a} + \mathbf{b} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle.
\]
If \( \mathbf{a} = \langle x, y, z \rangle \) and \( t \) is a scalar (a real number), then
\[
 t \mathbf{a} = \langle tx, ty, tz \rangle.
\]
The **length** (or **magnitude**) of the vector \( \mathbf{a} = \langle x, y, z \rangle \) is
\[
|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}.
\]
The **zero vector** is the vector \( \mathbf{0} = \langle 0, 0, 0 \rangle \).
The zero vector is the only vector that has no length or direction. Any single point is a representative of the zero vector.
If \( \mathbf{a} \) is a non–zero vector, then the **unit vector** in the direction of \( \mathbf{a} \) is
\[
\frac{1}{|\mathbf{a}|} \mathbf{a}.
\]
This vector has length 1 and points in the same direction as \( \mathbf{a} \).
If \( \mathbf{a} = \langle x, y, z \rangle \), then the **additive inverse** (or the **negative**) of \( \mathbf{a} \) is defined to be the vector

\[
-\mathbf{a} = \langle -x, -y, -z \rangle.
\]

Note that \(-\mathbf{a} = -1 \mathbf{a}\).
Example

Let \( \mathbf{a} = \langle 4, 3 \rangle \) and \( \mathbf{b} = \langle -2, 5 \rangle \).
(These are vectors in \( \mathbb{R}^2 \).)

1) Find the vector \( \mathbf{a} + \mathbf{b} \).
2) Find the vector \( \mathbf{a} - \mathbf{b} \).
3) Find the vector \( -2\mathbf{a} \).
4) Find the magnitude of \( \mathbf{a} \).
5) Find the unit vector in the direction of \( \mathbf{a} \).
Basic Algebraic Properties of Vectors

If \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) are vectors in \( \mathbb{R}^n \) and \( c \) and \( d \) are scalars, then

1) \( \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \)
2) \( \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \)
3) \( \mathbf{a} + \mathbf{0} = \mathbf{a} \)
4) \( \mathbf{a} + (-\mathbf{a}) = \mathbf{0} \)
5) \( c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b} \)
6) \( (c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} \)
7) \( (cd)\mathbf{a} = c(d\mathbf{a}) \)
8) \( 1\mathbf{a} = \mathbf{a} \).
Standard Basis Vectors

The standard basis vectors in $\mathbb{R}^2$ are

\[ \mathbf{i} = \langle 1, 0 \rangle \text{ and } \mathbf{j} = \langle 0, 1 \rangle. \]

If $\mathbf{a} = \langle x, y \rangle$ is any vector in $\mathbb{R}^2$, then

\[ \mathbf{a} = x\mathbf{i} + y\mathbf{j}. \]

Likewise, the standard basis vectors in $\mathbb{R}^3$ are

\[ \mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \text{ and } \mathbf{k} = \langle 0, 0, 1 \rangle. \]

If $\mathbf{a} = \langle x, y, z \rangle$ is any vector in $\mathbb{R}^3$, then

\[ \mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \]
A force acting on an object is a vector quantity because it has both a magnitude and a direction. Force is measured in Newtons or pounds.

The resultant force acting on an object is the vector sum of all of the forces acting on the object.

\[ \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \]
Example

A 100 pound weight hangs from two wires as pictured. Find the tensions (forces) exerted by each wire and find the magnitudes of these tensions.