Catapult Problem

Problem 27 in Section 10.4 of the Stewart textbook
A medieval city has the shape of a square and is protected by walls with length 500 meters and height 15 meters. You are the commander of an attacking army and the closest you can get to the wall is 100 meters. Your plan is to set fire to the city by catapulting heated rocks over the wall with an initial speed of 80 meters per second. At what range of angles should you tell your men to set the catapult? Assume that you will be firing from exactly 100 meters from the wall and that you will be firing perpendicular to the wall. We also assume that the walls of the city are impervious to heated rocks.

**SOLUTION**

First we draw a picture of the situation that shows what happens at various different firing angles. In this picture, trajectories of shots that hit the city are red and trajectories that miss are blue.
We can see that in the range of angles $(0^\circ, 90^\circ)$, there is
1) an interval of angles $(0^\circ, \alpha_1)$ for which the projectile hits the front wall (and hence misses its target)
2) an interval $(\alpha_1, \alpha_2)$ for which the projectile goes over the front wall and either lands directly in the city or perhaps hits the back wall of the city (and thus lands in the city)
3) an interval $(\alpha_2, \alpha_3)$ for which the projectile passes over both walls (and hence misses its target)
4) another interval $(\alpha_3, \alpha_4)$ for which the projectile goes over the front wall and either lands directly in the city or perhaps hits the back wall of the city (and thus lands in the city)
5) an interval $(\alpha_4, 90^\circ)$ for which the projectile is shot so high that it lands before reaching the front wall.
The equations for the path of the projectile are

\[ x = 80 \cos(\alpha) t \]
\[ y = 80 \sin(\alpha) t - \frac{1}{2} gt^2 \]

where \( \alpha \) is the angle of firing and \( g = 9.8 \text{ m/s}^2 \).

To find the angles \( \alpha_1 \) and \( \alpha_4 \) for which the projectile would hit the top of the front wall, we set \( x = 100 \) and \( y = 15 \). This gives

\[ 15 = 80 \sin(\alpha) \left( \frac{100}{80 \cos(\alpha)} \right) - \frac{1}{2} g \left( \frac{100}{80 \cos(\alpha)} \right)^2 \]

or

\[ 100 \tan(\alpha) - 7.65625 \sec^2(\alpha) = 15 \]
Using the identity \( \sec^2(\alpha) = 1 + \tan^2(\alpha) \), we obtain

\[
100 \tan(\alpha) - 7.65625(1 + \tan^2(\alpha)) = 15
\]

which gives

\[
7.65625 \tan^2(\alpha) - 100 \tan(\alpha) + 22.65625 = 0.
\]

Using the quadratic formula, we obtain

\[
\tan(\alpha) = \frac{100 \pm \sqrt{9306.152344}}{15.3125}
\]

This gives \( \tan(\alpha) = 12.831 \) or \( \tan(\alpha) = 0.23064 \) from which we obtain \( \alpha = 85.54^\circ \) or \( \alpha = 12.99^\circ \).

Therefore \( \alpha_1 = 12.99^\circ \) and \( \alpha_4 = 85.54^\circ \).
To find the angles $\alpha_2$ and $\alpha_3$ for which the projectile would hit the top of the back wall, we set $x = 600$ and $y = 15$. This gives

$$15 = 80 \sin(\alpha) \left( \frac{600}{80 \cos(\alpha)} \right) - \frac{1}{2} g \left( \frac{600}{80 \cos(\alpha)} \right)^2$$

or

$$600 \tan(\alpha) - 275.625 \sec^2(\alpha) = 15$$

which gives

$$600 \tan(\alpha) - 275.625(1 + \tan^2(\alpha)) = 15$$

which gives

$$275.625 \tan^2(\alpha) - 600 \tan(\alpha) + 290.625 = 0.$$

Using the quadratic formula, we obtain

$$\tan(\alpha) = \frac{600 \pm \sqrt{39585.9375}}{551.25}$$

This gives $\tan(\alpha) = 1.4494$ or $\tan(\alpha) = 0.72751$ from which we obtain $\alpha = 55.40^\circ$ or $\alpha = 36.04^\circ$. Therefore $\alpha_2 = 36.04^\circ$ and $\alpha_3 = 55.40^\circ$. 
Conclusion