1 Work, Flow, and Flux

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1.1 Work

If

\[ C : \mathbf{r} (t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \]

\[ a \leq t \leq b \]

is a smooth curve that lies in some region, \( D \), in which the continuous force field

\[ \mathbf{F} (x, y, z) = M(x, y, z) \mathbf{i} + N(x, y, z) \mathbf{j} + P(x, y, z) \mathbf{k} \]

is defined, then the work done by \( \mathbf{F} \) on a particle that traverses the curve \( C \) is

\[ \text{Work} = \int_{C} \mathbf{F} (x, y, z) \cdot \mathbf{T} (x, y, z) \, ds \]

where \( \mathbf{T} (x, y, z) \) is the unit tangent vector to the curve at the point \((x, y, z)\).

The work integral can be computed as

\[ \text{Work} = \int_{a}^{b} \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt \]

\[ = \int_{a}^{b} \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) \, dt \]

where \( M = M(x(t), y(t), z(t)) \), etc.

1.1.1 Example

Find the work done by the force field

\[ \mathbf{F}(x, y) = -10\mathbf{j} \]

in moving an object from the point \((0, 5)\) to the point \((5, 0)\) along the circle of radius 5 centered at the origin.
Solution: The given curve has parameterization
\[ C : r (t) = 5 \cos (t) \mathbf{i} + 5 \sin (t) \mathbf{j} \]
\[ \frac{\pi}{2} \geq t \geq 0 \]
so the work done by \( \mathbf{F} \) is
\[
\text{Work} = \int_{a}^{b} \left( M \frac{dx}{dt} + N \frac{dy}{dt} \right) \, dt \\
= \int_{\pi/2}^{0} \left( (0) (-5 \sin (t)) + (-10) (5 \cos (t)) \right) \, dt \\
= - \int_{\pi/2}^{0} 50 \cos (t) \, dt \\
= 50.
\]

1.2 Flow

If
\[ C : r (t) = x (t) \mathbf{i} + y (t) \mathbf{j} \]
\[ a \leq t \leq b \]
is a smooth curve that lies in some region, \( D \), in which the continuous velocity field
\[ \mathbf{V} (x, y) = M (x, y) \mathbf{i} + N (x, y) \mathbf{j} \]
is defined, then the flow induced by \( \mathbf{V} \) on a particle that traverses the curve \( C \) is
\[ \text{Flow along } C = \int_{C} \mathbf{V} (x, y) \cdot \mathbf{T} (x, y) \, ds \]
where \( \mathbf{T} (x, y) \) is the unit tangent vector to the curve at the point \((x, y)\).

The flow integral can be computed as
\[
\text{Flow along } C = \int_{a}^{b} \mathbf{V} (x (t), y (t)) \cdot \mathbf{r}' (t) \, dt \\
= \int_{a}^{b} \left( M \frac{dx}{dt} + N \frac{dy}{dt} \right) \, dt
\]
where \( M = M (x (t), y (t)) \), etc.
1.3 Flux

If

$$C : r(t) = x(t)i + y(t)j$$

$$a \leq t \leq b$$

is a smooth curve that lies in some region, $D$, in which the continuous velocity field

$$V(x, y) = M(x, y)i + N(x, y)j$$

is defined, then the flux induced by $V$ across the curve $C$ is

$$\text{Flux across } C = \int_C V(x, y) \cdot n(x, y) \, ds$$

where $n(x, y)$ is a unit normal vector to the curve $C$ at the point $(x, y)$. To be specific in determining one of two possible directions for $n(x, y)$, we adopt the convention to take

$$n = T(x, y) \times k$$

$$= \frac{1}{|r'(t)|} \left( \frac{dx}{dt}i + \frac{dy}{dt}j \right) \times k$$

$$= \frac{1}{|r'(t)|} \left( \frac{dy}{dt}i - \frac{dx}{dt}j \right)$$

and this leads us to conclude that flux across $C$ is

$$\text{Flux across } C = \int_C V(x, y) \cdot n(x, y) \, ds$$

$$= \int_a^b \left( M \frac{dy}{dt} - N \frac{dx}{dt} \right) \, dt.$$

1.4 Examples of Flow and Flux

1.4.1 Example 1

Let $V(x, y) = 5i$ and let $C$ be the curve $y = 0$, $0 \leq x \leq 4$. (Note that the flow is parallel to the curve $C$.) Find the flow along $C$ and the flux across $C$. 
**Solution:** The curve $C$ can be parameterized as

$$C : \mathbf{r} (t) = ti$$

$$0 \leq t \leq 4.$$ 

Thus

Flow along $C = \int_{0}^{4} ((0) (1) + (5) (0)) \, dt = 20$

and

Flux across $C = \int_{0}^{4} ((5) (0) - (0) (1)) \, dt = 0$.

It makes sense that the flux is $0$ because the velocity field is always parallel to the curve $C$.

**1.4.2 Example 2**

Let $\mathbf{V} (x, y) = 5\mathbf{j}$ and let $C$ be the curve $y = 0, 0 \leq x \leq 4$. (Note that the flow is perpendicular to the curve $C$.) Find the flow along $C$ and the flux across $C$.

**Solution:** The curve $C$ can be parameterized as

$$C : \mathbf{r} (t) = ti$$

$$0 \leq t \leq 4.$$ 

Thus

Flow along $C = \int_{0}^{4} ((0) (1) + (5) (0)) \, dt = 0$

and

Flux across $C = \int_{0}^{4} ((0) (0) - (5) (1)) \, dt = -20$.

Note that the flow along $C$ is $0$ because the velocity field is perpendicular to $C$. Also note that the flux is negative because the unit normal vector $\mathbf{n} (x, y) = \mathbf{T} (x, y) \times \mathbf{k} = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$ point in the opposite direction of the velocity field at all points on $C$. 

1.4.3 Example 3

Let $V(x, y) = -yi + xj$ and let $C$ be the curve

$$C : \mathbf{r}(t) = \cos(t) \mathbf{i} + \sin(t) \mathbf{j}$$

$$0 \leq t \leq 2\pi.$$ 

Find the flow along $C$ and the flux across $C$.

**Solution:**

Flow along $C$ = \[\int_{0}^{2\pi} ((-\sin(t)) (-\sin(t)) + (\cos(t))(\cos(t))) \, dt\]

= \[\int_{0}^{2\pi} 1 \, dt\]

= \[2\pi\]

and

Flux across $C$ = \[\int_{0}^{4} ((-\sin(t))(\cos(t)) - (\cos(t))(-\sin(t))) \, dt\]

= \[\int_{0}^{4} 0 \, dt\]

= \[0\].

In this example, the flow is always tangent to the path of motion. Thus the flux is 0.

1.4.4 Example 4

For the velocity field $V(x, y) = 5i$ and the curve $y = x^2$, find the flow and flux for the pieces of the curve corresponding to

a) $0 \leq x \leq 1$

b) $1 \leq x \leq 2$

c) $-1 \leq x \leq 1$

Convince yourself that the answers make sense from a physical perspective.

**Solution:**

a) For $0 \leq x \leq 1$, the curve we are considering is

$$C : \mathbf{r}(t) = ti + t^2 j$$

$$0 \leq t \leq 1.$$
We have

Flow along $C = \int_0^1 ((5)(1) + (0)(2t)) \, dt = \int_0^1 5 \, dt = 5$

and

Flux across $C = \int_0^1 ((5)(2t) - (0)(1)) \, dt = \int_0^1 10t \, dt = 5$

b) For $1 \leq x \leq 2$, the curve we are considering is

$C : \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}$

$1 \leq t \leq 2$.

We have

Flow along $C = \int_1^2 ((5)(1) + (0)(2t)) \, dt = \int_1^2 5 \, dt = 5$

and

Flux across $C = \int_1^2 ((5)(2t) - (0)(1)) \, dt = \int_1^2 10t \, dt = 15$

a) For $-1 \leq x \leq 1$, the curve we are considering is

$C : \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}$

$-1 \leq t \leq 1$. 
We have

Flow along $C = \int_{-1}^{1} ((5)(1) + (0)(2t)) \, dt$
\[= \int_{-1}^{1} 5 \, dt \]
\[= 10 \]

and

Flux across $C = \int_{-1}^{1} ((5)(2t) - (0)(1)) \, dt$
\[= \int_{-1}^{1} 10t \, dt \]
\[= 0.\]