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Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. If two objects travel through space along two different curves, it’s often important to know whether the two objects will collide. The two paths of motion might intersect, but we need to know whether the two objects are in the same place at the same time. Suppose that the trajectories of two different particles are given, respectively, by the vector–valued functions

\[ r_1(t) = \langle 3, \sqrt{3}t, 6t^2 + \left(8\sqrt{3} - 12\right)t + 14 - 8\sqrt{3} \rangle \]

and

\[ r_2(t) = \langle 3t^2, \sqrt{3}t, 8 \rangle. \]

Do the two particles collide? If so, at what time do they collide?

Solution: In order for the particles to be in the same place at the same time, there must be some time \( t \) such that all of the equations

\begin{align*}
3t^2 &= 3 \\
\sqrt{3}t &= \sqrt{3}t \\
8 &= 6t^2 + \left(8\sqrt{3} - 12\right)t + 14 - 8\sqrt{3}
\end{align*}

are satisfied.

The first of these equations is satisfied for \( t = 1 \) and so is the second. Let us see if \( t = 1 \) satisfies the third equation:

\[ 6(1)^2 + \left(8\sqrt{3} - 12\right)(1) + 14 - 8\sqrt{3} = 6 + 8\sqrt{3} - 12 + 14 - 8\sqrt{3} = 8. \]

Thus the particles do collide at time \( t = 1 \).

2. Find the unit tangent vector to the curve

\[ r(t) = \langle 3t^2, \sqrt{3}t, 8 \rangle \]

at the point on the curve corresponding to \( t = 1 \).
Solution: Since

\[ \mathbf{r}'(t) = \langle 6t, \sqrt{3}, 0 \rangle \]

and

\[ |\mathbf{r}'(t)| = \sqrt{36t^2 + 3}, \]

the unit tangent vector (at any point on the curve) is

\[ \mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}'(t) = \left\langle \frac{6t}{\sqrt{36t^2 + 3}}, \frac{\sqrt{3}}{\sqrt{36t^2 + 3}}, 0 \right\rangle. \]

(It is easy to observe that \( |\mathbf{T}(t)| = 1 \) for all \( t \).) The unit tangent vector at the point on the curve corresponding to \( t = 1 \) is

\[ \mathbf{T}(1) = \left\langle \frac{6}{\sqrt{39}}, \frac{\sqrt{3}}{\sqrt{39}}, 0 \right\rangle. \]

3. Find the curvature of the curve

\[ \mathbf{r}(t) = \langle e^{t} \cos(t), e^{t} \sin(t), t \rangle \]

at the point \((1, 0, 0)\). (You must, of course, include all calculations in detail.)

Solution: First we compute

\[ \mathbf{r}'(t) = \langle -e^{t} \sin(t) + e^{t} \cos(t), e^{t} \cos(t) + e^{t} \sin(t), 1 \rangle \]

\[ = \langle e^{t} (\cos(t) - \sin(t)), e^{t} (\cos(t) + \sin(t)), 1 \rangle \]

\[ \mathbf{r}''(t) \]

\[ = \langle e^{t} (-\sin(t) - \cos(t)) + e^{t} (\cos(t) - \sin(t)), e^{t} (-\sin(t) + \cos(t)) + e^{t} (\cos(t) + \sin(t)), 0 \rangle \]

\[ = \langle -2e^{t} \sin(t), 2e^{t} \cos(t), 0 \rangle \]

and

\[ |\mathbf{r}'(t)| = \sqrt{(e^{t} (\cos(t) - \sin(t)))^2 + (e^{t} (\cos(t) + \sin(t)))^2 + (1)^2} \]

\[ = \sqrt{e^{2t} (1 - 2 \sin(t) \cos(t)) + e^{2t} (1 + 2 \sin(t) \cos(t)) + 1} \]

\[ = \sqrt{2e^{2t} + 1}. \]

At \( t = 0 \) (which corresponds to the point \((1, 0, 0)\)), we have

\[ \mathbf{r}'(0) = \langle 1, 1, 1 \rangle \]

\[ \mathbf{r}''(0) = \langle 0, 2, 0 \rangle \]

\[ |\mathbf{r}'(0)| = \sqrt{3}. \]
Also
\[ r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2i + 2k \]

and hence
\[ |r'(0) \times r''(0)| = \sqrt{8}. \]

The curvature of the curve at the point \((1, 0, 0)\) is
\[ \kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{\sqrt{8}}{(\sqrt{3})^3} = \frac{8^{1/2}}{3^{3/2}} = \frac{1}{3} \sqrt{\frac{8}{3}}. \]

4. Given that the acceleration function of a moving particle is
\[ a(t) = 2i + 6tj + 12t^2k, \]
the initial velocity of the particle is
\[ v(0) = i, \]
and the initial position of the particle is
\[ r(0) = j - k, \]
find the velocity function and the position function of the particle. (All details must be included in your solution.)

**Solution:** The velocity function is
\[ v(t) = 2ti + 3t^2j + 4t^3k + C \]
where \(C\) is a constant vector. Since we must have
\[ 2(0)i + 3(0)^2j + 4(0)^3k + C = i, \]
we conclude that \(C = i\). Thus, the velocity function is
\[ v(t) = (2t + 1)i + 3t^2j + 4t^3k. \]
The position function is
\[ r(t) = (t^2 + t)i + t^3j + t^4k + C \]
where \(C\) is a constant vector. Since we must have
\[ ((0)^2 + 0)i + (0)^3j + (0)^4k + C = j - k, \]
we conclude that \(C = j - k\). Thus, the position function of the particle is
\[ r(t) = (t^2 + t)i + (t^3 + 1)j + (t^4 - 1)k. \]
5. Match the equations given in a–e with the pictures. (Write each parametric formula next to the graph that is described by the formula.)

(a) \( r(u,v) = (u^2 + 1, v^3 + 1, u + v), \quad -1 \leq u \leq 1, \quad -1 \leq v \leq 1 \)

(b) \( r(u,v) = (u + v, u^2, v^2), \quad -1 \leq u \leq 1, \quad -1 \leq v \leq 1 \)

(c) \( r(u,v) = (2 \sin(u), 3 \cos(u), v), \quad 0 \leq u \leq 2\pi, \quad -1 \leq v \leq 1 \)

(d) \( r(u,v) = (u^3, u \sin(v), u \cos(v)), \quad -1 \leq u \leq 1, \quad 0 \leq v \leq 2\pi \)

(e) \( r(u,v) = (1 - \cos(u)) \sin(v), u, (u - \sin(u)) \cos(v)), \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq \frac{2\pi}{2\pi} \)

\[ r(u,v) = (u^3, u \sin(v), u \cos(v)) \]

\[ r(u,v) = (u^2 + 1, v^3 + 1, u + v) \]
\[ r(u, v) = (2 \sin(u), 3 \cos(u), v) \]
\[ r(u, v) = \langle u + v, u^2, v^2 \rangle \]