1. Match each of the functions (a and b) with its graph (labelled A and B) and with its contour map (labelled I and II). Write a brief explanation (a few sentences) explaining some of the reasons for your answers.

(a) \( z = \sin (x - y) \) matches A and I.
(b) \( z = \sin (x) - \sin (y) \) matches B and II.
2. Find the limit

\[ \lim_{(x,y) \to (0,0)} \frac{6x^3y}{2x^4 + y^4} \]

if it exists or show that the limit does not exist. (Make sure that your reasoning is clearly explained.)
Solution: If we let \((x, y) \to (0, 0)\) along the line \(y = x\), then we have

\[
\lim_{(x, y) \to (0, 0)} \frac{6x^3y}{2x^4 + y^4} = \lim_{x \to 0} \frac{6x^3}{2x^4 + x^4} = \lim_{x \to 0} \frac{6x^4}{3x^4} = \lim_{x \to 0} 2 = 2.
\]

If we let \((x, y) \to (0, 0)\) along the line \(y = -x\), then we have

\[
\lim_{(x, y) \to (0, 0)} \frac{6x^3y}{2x^4 + y^4} = \lim_{x \to 0} \frac{6x^3(-x)}{2x^4 + (-x)^4} = \lim_{x \to 0} \frac{-6x^4}{3x^4} = \lim_{x \to 0} (-2) = -2.
\]

We conclude that the limit in question does not exist.

3. Find the first and second partial derivatives of the function

\(f(x, y) = 3x - 2y^4\).

(Note: There are two first partial derivatives and four second partial derivatives.)

Solution:

\[
f_x = 3, \quad f_y = -8y^3, \quad f_{xx} = 0, \quad f_{xy} = 0, \quad f_{yx} = 0, \quad f_{yy} = -24y^2.
\]

4. Find the linearization, \(L(x, y)\), of the function

\(f(x, y) = \frac{x}{y}\)

at the point \((6, 3)\).

Solution:

\[
f_x = \frac{1}{y}
\]
and
\[ f_y = \frac{-x}{y^2} \]
so
\[ f_x (6, 3) = \frac{1}{3} \]
and
\[ f_y (6, 3) = \frac{-6}{3^3} = \frac{-2}{9}. \]
Thus the linearization of \( f \) at \((6, 3)\) is
\[
L (x, y) = f (6, 3) + f_x (6, 3) (x - 6) + f_y (6, 3) (y - 3)
\]
\[
= 2 + \frac{1}{3} (x - 6) - \frac{2}{9} (y - 3).
\]

5. For the function
\[ z = x^2 + xy + y^2 \]
where
\[ x = s + t \]
and
\[ y = st, \]
use the Chain Rule to find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).
**Solution:**
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
\[
= (2x + y) (1) + (x + 2y) (t)
\]
\[
= (2 (s + t) + st) + ((s + t) + 2st) (t)
\]
\[
= 2s + 2t + st + st + t^2 + 2st^2
\]
\[
= 2s + 2t + 2st + t^2 + 2st^2
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]
\[
= (2x + y) (1) + (x + 2y) (s)
\]
\[
= (2 (s + t) + st) + ((s + t) + 2st) (s)
\]
\[
= 2s + 2t + st + s^2 + ts + 2s^2 t
\]
\[
= 2s + 2t + 2st + s^2 + 2s^2 t.
\]

6. Near a buoy, the depth of a lake at the point with coordinates \((x, y)\) is
\[ z = 200 + 0.02x^2 - 0.001y^3 \]
where \( x, y, \) and \( z \) are measured in meters. A fisherman in a small boat starts at the point \((80, 60)\) and moves toward the buoy (which is located at the point \((0, 0)\)). Is the
water under the boat getting deeper or shallower when he departs? (Explain in detail. Your explanation should include some calculations and a written explanation of your reasoning.)

**Solution:** We need to find the directional derivative of the function

\[ f(x, y) = 200 + 0.02x^2 - 0.001y^3 \]

at the point \((80, 60)\) in the direction of the vector \(\mathbf{v} = (-80, -60)\) (since this is a vector that points from the point \((80, 60)\) to the origin).

Since

\[ |\mathbf{v}| = \sqrt{(-80)^2 + (-60)^2} = 100, \]

the unit vector in the direction of \(\mathbf{v}\) is

\[ \mathbf{u} = \left\langle \frac{-80}{100}, \frac{-60}{100} \right\rangle = (-0.8, -0.6). \]

Also

\[ \nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} = 0.04x \mathbf{i} - 0.003y^2 \mathbf{j} \]

so

\[ \nabla f(80, 60) = 0.04(80) \mathbf{i} - 0.003(60)^2 \mathbf{j} = 3.2 \mathbf{i} - 10.8 \mathbf{j} \]

We now compute:

\[ D_v f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} \]

\[ = (3.2 \mathbf{i} - 10.8 \mathbf{j}) \cdot (-0.8 \mathbf{i} - 0.6 \mathbf{j}) \]

\[ = (3.2)(-0.8) + (-10.8)(-0.6) \]

\[ = 3.92. \]

Since this directional derivative is positive, we conclude that the depth of the lake is increasing at the rate of 3.92 meters per meter as the boat moves away from the point \((80, 60)\) in the direction of the buoy.

7. Find all points at which the function

\[ f(x, y) = e^{4y-x^2-y^2} \]

has a local maximum, a local minimum, or a saddle point. Be sure to include all details of your solution, including how you determine whether each critical point that you find corresponds to a local maximum, a local minimum, or a saddle point.

**Solution:** First we find the critical points of \(f\):

\[ f_x = -2xe^{4y-x^2-y^2} \]

and

\[ f_y = (4 - 2y)e^{4y-x^2-y^2} \]

so the only critical point of \(f\) is \((0, 2)\). The value of \(f\) at this critical point is

\[ f(0, 2) = e^{4(2)-(0)^2-(2)^2} = e^4 \approx 54.6. \]
To classify this critical point, we must compute the second derivatives of $f$:

$$f_{xx} = 4x^2 e^{4y-x^2-y^2} - 2e^{4y-x^2-y^2} = 2e^{4y-x^2-y^2} (2x^2 - 1)$$

$$f_{yy} = (4 - 2y)^2 e^{4y-x^2-y^2} - 2e^{4y-x^2-y^2}$$

$$= (16 - 16y + 4y^2 - 2) e^{4y-x^2-y^2}$$

$$= (4y^2 - 16y + 14) e^{4y-x^2-y^2}$$

$$= 2e^{4y-x^2-y^2} (2y^2 - 8y + 7)$$

$$f_{xy} = -2x (4 - 2y) e^{4y-x^2-y^2}$$

and observe that

$$D = f_{xx} (0, 2) f_{yy} (0, 2) - (f_{xy} (0, 2))^2$$

$$= (-2e^4) (-2e^4) - (0^2)$$

$$= 4e^4$$

which is a positive number, and

$$f_{xx} (0, 2) = -2e^4$$

which is a negative number.

By the Second Derivative Test, $f$ has a local maximum value of $e^4$ that occurs at the point $(0, 2)$. A graph of $f$ is shown below.