1. Match each of the functions (a and b) with its graph (labelled A and B) and with its contour map (labelled I and II). Write a brief explanation (a few sentences) explaining some of the reasons for your answers.

(a) \( z = \frac{x-y}{1+x^2+y^2} \) matches A and II.
(b) \( z = \frac{1}{1+x^2+y^2} \) matches B and I.

The graphs are on the next two pages. Write your explanation for your choices here:
2. Explain why the function
\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]
is not continuous at the point (0, 0). (Be detailed in your answer. Your answer should include at least one calculation and a written explanation.)
Solution: If we let \((x, y) \to (0, 0)\) along the line \(y = x\), then we have

\[
\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y=x} \frac{xy}{x^2 + xy + y^2}
\]
\[
= \lim_{y=x} \frac{xx}{x^2 + xx + x^2}
\]
\[
= \lim_{x\to0} \frac{x^2}{3x^2}
\]
\[
= \lim_{x\to0} \frac{1}{3}
\]
\[
= \frac{1}{3}.
\]

This tells us, in particular, that

\[
\lim_{(x,y)\to(0,0)} f(x, y) \neq 0.
\]

(If the above limit exists, it must be equal to \(1/3\).)

Since

\[
\lim_{(x,y)\to(0,0)} f(x, y) \neq f(0, 0),
\]
then \(f\) is not continuous at the point \((0, 0)\).

3. For the implicitly–defined surface

\[
x^2 + y^2 + z^2 = 3xyz,
\]
use implicit differentiation to find \(\frac{\partial z}{\partial x}\).

Solution: There are two ways to do this. One way is to define

\[
F(x, y, z) = x^2 + y^2 + z^2 - 3xyz
\]

and then to use the formula

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.
\]

Since

\[
F_z = 2z - 3xy
\]
and

\[
F_x = 2x - 3yz,
\]
we obtain

\[
\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z - 3xy}.
\]

Another way to do the problem is to apply the operator \(\partial / \partial x\) to both sides of the equation

\[
x^2 + y^2 + z^2 = 3xyz
\]
(bearing in mind that \( \partial x/\partial x = 1 \) and \( \partial y/\partial x = 0 \)) to obtain

\[
\frac{\partial}{\partial x} (x^2 + y^2 + z^2) = \frac{\partial}{\partial x} (3xyz)
\]

and thus

\[
2x + 2z \frac{\partial z}{\partial x} = 3xy \frac{\partial z}{\partial x} + 3yz
\]

which gives

\[
(2z - 3xy) \frac{\partial z}{\partial x} = 3yz - 2x
\]

and finally

\[
\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z - 3xy}.
\]

4. Find the linearization, \( L(x, y) \), of the function

\[
f(x, y) = xy e^x
\]

at the point \((\ln(2), 0)\).

**Solution:**

\[
\frac{\partial f}{\partial x} = xy e^x + ye^x = ye^x (x + 1)
\]

and

\[
\frac{\partial f}{\partial y} = xe^x
\]

so

\[
\frac{\partial f}{\partial x} (\ln(2), 0) = 0
\]

and

\[
\frac{\partial f}{\partial y} (\ln(2), 0) = 2 \ln(2) = \ln(4).
\]

The linearization is thus

\[
L(x, y) = f(\ln(2), 0) + \frac{\partial f}{\partial x} (\ln(2), 0) (x - \ln(2)) + \frac{\partial f}{\partial y} (\ln(2), 0) (y - 0)
\]

\[
= 0 + 0 (x - \ln(2)) + \ln(4) (y - 0)
\]

\[
= \ln(4) y.
\]

5. For the function

\[
M = xe^{y-z^2}
\]

where

\[
x = 2uv,
\]

\[
y = u - v,
\]

and

\[
z = u + v,
\]

use the Chain Rule to find \( \partial M/\partial v \).
6. Find the maximum rate of change of the function

\[ f(x, y) = ye^{-x} + xe^{-y} \]

at the point \((0, 0)\) and the direction in which this maximum rate of change occurs. (Be sure to include all details of your solution and be sure to write statements in complete sentences.)

**Solution:** The gradient vector of \(f\) is

\[ \nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} = (-ye^{-x} + e^{-y}) \mathbf{i} + (e^{-x} - xe^{-y}) \mathbf{j}. \]

At the point \((0, 0)\) we have

\[ \nabla f(0, 0) = \mathbf{i} + \mathbf{j}. \]

This tells us that, at the point \((0, 0)\), the function \(f\) increases most rapidly in the direction that the vector \(\mathbf{i} + \mathbf{j}\) points. The actual rate of change of \(f\) in this direction (which is the maximum rate of change of \(f\) in any direction) is

\[ |\nabla f(0, 0)| = |\mathbf{i} + \mathbf{j}| = \sqrt{2}. \]

A graph of \(f\) near the origin is shown below.

7. Find the absolute minimum and maximum values (and where they occur) of the function

\[ f(x, y) = 4x + 6y - x^2 - y^2 \]

on the rectangular domain

\[ D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 5\}. \]

(Be detailed. Include a picture of the domain, all relevant calculations, and some written explanations that accompany your calculations.)

**Solution:** In order to find the absolute maximum and minimum and where they occur, we must first find the critical points of \(f:\)

\[ f_x = 4 - 2x \]
\[ f_y = 6 - 2y \]

tells us that the only critical point of \(f\) is \((2, 3)\). This critical point does lie in the domain \(D\), so let us record the fact that

\[ f(2, 3) = 4(2) + 6(3) - (2)^2 - (3)^2 = 13. \]
Now let us evaluate $f$ along each of the boundary curves of $D$:

Along the boundary curve $x = 0$, the function under consideration is

$$f(0, y) = g(y) = 6y - y^2 = y(6 - y), \quad 0 \leq y \leq 5.$$  

The graph of $g$ is a parabola (shown below).

On the interval $0 \leq y \leq 5$, $g$ achieves its absolute minimum value of 0 at $y = 0$ and achieves its absolute maximum value of 9 at $y = 3$. We record

$$f(0, 0) = 0$$
$$f(0, 3) = 9.$$

The analysis over the other three boundaries of $D$ is similar.

Along the boundary curve $y = 5$, the function under consideration is

$$f(x, 5) = g(x) = 4x + 30 - x^2 - 25 = -(x - 5)(x + 1), \quad 0 \leq x \leq 4.$$  

On the interval $0 \leq x \leq 4$, $g$ achieves its absolute minimum value of 5 at $x = 0$ and achieves its absolute maximum value of 8 at $x = 3$. We record

$$f(0, 5) = 5$$
$$f(3, 5) = 8.$$

Along the boundary curve $x = 4$, the function under consideration is

$$f(4, y) = g(y) = 16 + 6y - 16 - y^2 = y(6 - y), \quad 0 \leq y \leq 5.$$  

On the interval $0 \leq y \leq 5$, $g$ achieves its absolute minimum value of 0 at $y = 0$ and achieves its absolute maximum value of 9 at $y = 3$. We record

$$f(4, 0) = 0$$
$$f(4, 3) = 9.$$
Along the boundary curve $y = 0$, the function under consideration is
\[ f(x, 0) = g(x) = 4x - x^2 = x(4 - x), \quad 0 \leq x \leq 4. \]

On the interval $0 \leq x \leq 4$, $g$ achieves its absolute minimum value of 0 at $x = 0$ and at $x = 4$ and achieves its absolute maximum value of 4 at $x = 2$. We record
\[ f(0, 0) = 0 \]
\[ f(4, 0) = 0 \]
\[ f(2, 0) = 4. \]

Finally, comparing all of the candidate values that we have computed, we conclude that, on the domain $D$, $f$ has an absolute maximum value of 13 and this value occurs only at the critical point $(2, 3)$, and we conclude that $f$ has an absolute minimum value of 0 (on the domain $D$) and that this minimum value occurs at the boundary points $(0, 0)$ and $(4, 0)$.

A graph of $f$ over the domain $D$ is shown below.

Graph of $f(x, y) = 4x + 6y - x^2 - y^2$