1. Match each of the functions (a and b) with its graph (labelled A and B) and with its contour map (labelled I and II). Write a brief explanation (a few sentences) explaining some of the reasons for your answers.

(a) \( z = e^x \cos(y) \) matches A and I.

(b) \( z = e^{(x-y^2)} \) matches B and II.
2. Use the Squeezing Principle to show that

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{x^2 + 2y^2} = 0. \]

**Solution:** For any \((x, y) \neq (0, 0)\), it is true that

\[ 0 < x^2 \leq x^2 + 2y^2 \]

and hence that

\[ 0 < \frac{x^2}{x^2 + 2y^2} \leq 1. \]
If we multiply all parts of this inequality by $\sin^2(y)$, we obtain

$$0 \leq \frac{x^2 \sin^2(y)}{x^2 + 2y^2} \leq \sin^2(y).$$

Since

$$\lim_{(x,y) \to (0,0)} 0 = 0$$

and

$$\lim_{(x,y) \to (0,0)} \sin^2(y) = 0,$$

the Squeezing Principle allows us to conclude that

$$\lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{x^2 + 2y^2} = 0.$$

3. For the function of three variables

$$f(x, y, z) = xy^2z^3 + 3yz,$$

find $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

Solution:

$$\frac{\partial f}{\partial x} = y^2z^3,$$

$$\frac{\partial f}{\partial y} = 2xyz^3 + 3z,$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 + 3y.$$

4. Find the linearization, $L(x, y)$, of the function

$$f(x, y) = \arctan(x + 2y)$$

at the point $(1,0)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (x + 2y)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{2}{1 + (x + 2y)^2}$$

so

$$\frac{\partial f}{\partial x} (1,0) = \frac{1}{1 + (1 + 2(0))^2} = \frac{1}{2}$$

and

$$\frac{\partial f}{\partial y} (1,0) = \frac{2}{1 + (1 + 2(0))^2} = 1.$$
The linearization is thus
\[ L(x, y) = f(1, 0) + \frac{\partial f}{\partial x}(1, 0)(x - 1) + \frac{\partial f}{\partial y}(1, 0)(y - 0) \]
\[ = \frac{\pi}{4} + \frac{1}{2}(x - 1) + y. \]

5. For the function
\[ w = xe^{y/z} \]
where
\[ x = t^2, \]
\[ y = 1 - t, \]
and
\[ z = 1 + 2t, \]
use the Chain Rule to find \( dw/dt \).

**Solution:**
\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \]
\[ = \left( e^{y/z} \right)(2t) + \left( \frac{x}{z} e^{y/z} \right)(-1) + \left( \frac{-xy}{z^2} e^{y/z} \right)(2) \]

6. Find the directional derivative of the function
\[ f(x, y, z) = x^2 + y^2 + z^2 \]
at the point \((2, 1, 3)\) in the direction of the origin. (Please be thorough and don’t omit any details. Include some written explanation – writing in complete sentences.)

**Solution:** First we note that the gradient vector of \( f \) is
\[ \nabla f(x, y, z) = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} = 2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k} \]
which tells us that
\[ \nabla f(2, 1, 3) = 4 \mathbf{i} + 2 \mathbf{j} + 6 \mathbf{k}. \]
A vector pointing from the point \((2, 1, 3)\) to the origin is \( \mathbf{v} = -2 \mathbf{i} - \mathbf{j} - 3 \mathbf{k} \) and since the length of this vector is
\[ |\mathbf{v}| = \sqrt{(-2)^2 + (-1)^2 + (-3)^2} = \sqrt{14}, \]
then the unit vector pointing in the same direction is
\[ \mathbf{u} = -\frac{2\sqrt{14}}{14} \mathbf{i} - \frac{\sqrt{14}}{14} \mathbf{j} - \frac{3\sqrt{14}}{14} \mathbf{k}. \]
Therefore
\[
D_v f (2, 1, 3) = \nabla f (2, 1, 3) \cdot u
\]
\[
= (4i + 2j + 6k) \cdot \left( -\frac{2\sqrt{14}}{14}i - \frac{\sqrt{14}}{14}j - \frac{3\sqrt{14}}{14}k \right)
\]
\[
= 4 \left( -\frac{2\sqrt{14}}{14} \right) + 2 \left( -\frac{\sqrt{14}}{14} \right) + 6 \left( -\frac{3\sqrt{14}}{14} \right)
\]
\[
= -2\sqrt{14}.
\]

7. Find the points on the surface
\[z^2 = xy + 9\]
that are closest to the origin. (Include all relevant details of your solution, including all relevant calculations and an accompanying written explanation.)

**Solution:** The square of the distance from any point \((x, y, z)\) to the origin \((0, 0, 0)\) is
\[f (x, y, z) = x^2 + y^2 + z^2.\]
If we are considering only points on the given surface, then the square of the distance from such a point to the origin is
\[x^2 + y^2 + xy + 9.\]
We need to minimize the function
\[g (x, y) = x^2 + y^2 + xy + 9.\]
First we compute
\[g_x = 2x + y\]
\[g_y = 2y + x.\]
The system of equations
\[2x + y = 0\]
\[2y + x = 0\]
has \((0, 0)\) as its only solution, so this is the only critical point of \(g\). Furthermore, since
\[g_{xx} = 2\]
\[g_{yy} = 2\]
\[g_{xy} = 1\]
we see that
\[D = g_{xx}g_{yy} - (g_{xy})^2 = 3 > 0\]
and
\[g_{xx} = 2 > 0\]
which, by the Second Derivative Test, tells us that $(0,0)$ corresponds to a local minimum of $g$.

Finally, observe that $(x,y) = (0,0)$ corresponds to both of the points $(0,0,3)$ and $(0,0,-3)$ on the surface $z^2 = xy + 9$. These two points are the two points on the surface that are closest to the origin.