Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Describe in words (writing complete sentences) the region of $\mathbb{R}^3$ represented by the inequality $x^2 + y^2 \leq 4$. (It is also not hard to draw this region in $\mathbb{R}^3$. Even I can do it!)

Answer: The equation $x^2 + y^2 = 4$ is an equation for the circular cylinder of radius 2 whose central axis is the $z$ axis. Therefore the inequality $x^2 + y^2 \leq 4$ describes the set of all points inside this cylinder along with all points on this cylinder. This cylinder is pictured below.

2. Two vectors, $\mathbf{a}$ and $\mathbf{b}$, are pictured below.
(a) Which of the vectors pictured below is \( a + b \)? Circle the correct choice.

(b) Which of the vectors pictured below is \( a - b \)? Circle the correct choice.

(c) Which of the vectors pictured below is \( b - a \)? Circle the correct choice.

(d) Is \( a \cdot b \) a positive number, a negative number, or 0? You must explain your answer (in sentence form) in order to get credit for this.

\[ \text{Answer: } a \cdot b \text{ is a positive number because the angle between } a \text{ and } b \text{ is acute.} \]

(e) Does the vector \( a \times b \) point into the page or out of the page? You must explain your answer (in sentence form) in order to get credit for this.

\[ \text{Answer: } a \times b \text{ points into the page by the right hand rule.} \]

3. The vectors

\[ a = -i - 5j - 4k \]

and

\[ b = -4i - j + k \]

are (circle the correct choice)

(a) parallel to each other.

(b) perpendicular to each other.

(c) neither parallel nor perpendicular to each other.

\[ \text{Explain the reason for your answer. } \] (Write in sentence form and include any relevant calculations in your explanation.)

\[ \text{Answer: } \text{Recall that } a \cdot b = |a||b| \cos (\theta) \]

where \( \theta \) is the angle between \( a \) and \( b \) \((0^\circ \leq \theta \leq 180^\circ)\). If \( a \) and \( b \) are parallel to each other, then either \( \theta = 0 \) (in which case \( a \cdot b = |a||b| \)) or \( \theta = 180^\circ \) (in which case
\[ \mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|. \] If \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular to each other, then \( \theta = 90^\circ \) (in which case \( \mathbf{a} \cdot \mathbf{b} = 0 \)).

Since
\[ \mathbf{a} \cdot \mathbf{b} = (-1)(-4) + (-5)(-1) + (-4)(1) = 5 \]
and
\[ |\mathbf{a}| = \sqrt{(-1)^2 + (-5)^2 + (-4)^2} = \sqrt{42} \]
and
\[ |\mathbf{b}| = \sqrt{(-4)^2 + (-1)^2 + (1)^2} = 3\sqrt{2}, \]
we see that
\[ |\mathbf{a}||\mathbf{b}| = 6\sqrt{21}. \]

Since \( \mathbf{a} \cdot \mathbf{b} \neq 0 \), we know that \( \mathbf{a} \) and \( \mathbf{b} \) are not perpendicular to each other. Also since \( \mathbf{a} \cdot \mathbf{b} \neq |\mathbf{a}||\mathbf{b}| \) and \( \mathbf{a} \cdot \mathbf{b} \neq -|\mathbf{a}||\mathbf{b}| \), we know that \( \mathbf{a} \) and \( \mathbf{b} \) are not parallel to each other. (In addition, since \( \mathbf{a} \cdot \mathbf{b} > 0 \), the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is acute.)

4. Suppose that \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) are vectors. Decide whether each of the following expressions is equal to a vector, a scalar, or is not meaningful. (Circle the correct choice.)

(a) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) (is a vector / is a scalar / is not meaningful).
(b) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) (is a vector / is a scalar / is not meaningful).
(c) \( \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) \) (is a vector / is a scalar / is not meaningful).
(d) \( (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c} \) (is a vector / is a scalar / is not meaningful).
(e) \( (\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{b} \cdot \mathbf{c}) \) (is a vector / is a scalar / is not meaningful).

5. Find parametric equations for the line, \( L_1 \), that passes through the point \((3, 3, -4)\) and is parallel to the line, \( L_2 \), with parametric equations
\[
\begin{align*}
x &= -1 - 4t \\
y &= 5 \\
z &= 2 \\
-\infty &< t < \infty
\end{align*}
\]

**Solution:** Since \( L_1 \) is parallel to \( L_2 \), then the vector \((-4, 0, 0)\), which is a direction vector of \( L_2 \), is also a direction vector of \( L_1 \). From this fact, we see that parametric equations for \( L_1 \) are
\[
\begin{align*}
x &= 3 - 4t \\
y &= 3 \\
z &= -4 \\
-\infty &< t < \infty.
\end{align*}
\]
6. Here are six equations labelled 1–6 and six surfaces labelled A–F. Match each equation with the surface that it describes.
1) \[ x^2 + y^2 - z^2 = -1 \]
2) \[ x^2 + y^2 - z^2 = 0 \]
3) \[ x^2 + y^2 - z = 0 \]
4) \[ x^2 + y^2 + z^2 = 1 \]
5) \[ x^2 + y^2 - z^2 = 1 \]
6) \[ x^2 - y^2 - z = 0 \]

Equation 1 matches surface __________.
Equation 2 matches surface __________.
Equation 3 matches surface __________.
Equation 4 matches surface __________.
Equation 5 matches surface __________.
Equation 6 matches surface __________.

7. For each of the following questions, you must show your calculations.

(a) The rectangular coordinates of point \( P \) are \((1, -1, 4)\). Find cylindrical coordinates for \( P \).

\[ \text{Solution:} \quad r^2 = (1)^2 + (-1)^2 = 2 \quad \text{so} \quad r = \sqrt{2}. \]
Also \( \tan(\theta) = -1/1 = -1 \) and the point \((1, -1)\) is in quadrant IV of the \( xy \) plane, so we can take \( \theta = -\pi/4 \). Thus cylindrical coordinates for \( P \) are \((\sqrt{2}, -\pi/4, 4)\).

(b) Spherical coordinates of point \( P \) are \((2, \pi/3, \pi/4)\). Find the rectangular coordinates for \( P \).

\[ \text{Solution:} \]
\[ x = \rho \cos(\theta) \sin(\phi) = 2 \cos \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \]
\[ y = \rho \sin(\theta) \sin(\phi) = 2 \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{6}}{2} \]
\[ z = \rho \cos(\phi) = 2 \cos \left( \frac{\pi}{4} \right) = \sqrt{2} \]

shows that the rectangular coordinates for \( P \) are \((\sqrt{2}/2, \sqrt{6}/2, \sqrt{2})\).

(c) The rectangular coordinates of point \( P \) are \((-1, 1, \sqrt{6})\). Find spherical coordinates for \( P \).

\[ \text{Solution:} \quad \text{Using} \ \rho^2 = (-1)^2 + (1)^2 + (\sqrt{6})^2 = 8, \ \text{we obtain} \ \rho = 2\sqrt{2}. \quad \text{Also} \]
\[ \cos(\phi) = \frac{z}{\rho} = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \]
and $z > 0$ so we can take $\phi = \pi/6$. In addition

$$\cos (\theta) = \frac{x}{\rho \sin (\phi)} = \frac{-1}{2\sqrt{2} \left(\frac{1}{2}\right)} = -\frac{\sqrt{2}}{2}$$

and the point $(-1, 1)$ is in Quadrant II of the $xy$ plane so we can take $\theta = 3\pi/4$. Spherical coordinates for $P$ are thus $(\sqrt{8}, 3\pi/4, \pi/6)$.

(d) Describe in words the surface whose equation is given in spherical coordinates by $\rho = 3$.

**Answer:** This surface is a sphere of radius 3 centered at the origin in $\mathbb{R}^3$.

(e) Write an equation in rectangular coordinates for the surface that is described in cylindrical coordinates by $z = r^2$. Which type of quadric surface is this? (ellipsoid, elliptic paraboloid, elliptic cone, hyperboloid of one sheet, hyperboloid of two sheets, or hyperbolic paraboloid)?

**Solution:** In rectangular coordinates, $z = r^2$ becomes $z = x^2 + y^2$. This is an elliptic paraboloid.