Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Draw a contour map of the function

\[ f(x, y) = e^{(y/x)} \]

showing several (at least five) level curves. You must show the algebra that you perform in order to determine what the level curves look like.

Solution: For any positive constant, \( K \), if we set

\[ e^{(y/x)} = K \]

we obtain

\[ \frac{y}{x} = \ln (K) \]

or

\[ y = \ln (K) x. \]

This is the equation of a line passing through the origin (in the \( xy \) plane) with slope \( \ln (K) \). As \( K \) ranges through the interval \((0, \infty)\), \( \ln (K) \) ranges through the interval \((-\infty, \infty)\). Thus the level curves of this function are all possible lines passing through the origin (except the line \( x = 0 \) on which \( f \) is not defined). Actually, since \( f \) is not defined at the point \((0,0)\), each of the level curves has a “hole” at the point \((0,0)\).

The level curves pictured below correspond to \( K = 0.1, K = 0, K = 10, K = e^5 \approx 148.41 \), and \( K = e^{10} \).
2. Explain why

\[ \lim_{(x,y) \to (0,0)} \frac{xy^4}{x^2 + y^8} \]

does not exist. (Be sure to write in sentences and include details.)

Explanation: If we let \((x, y)\) approach \((0, 0)\) along the line \(y = 0\), then we obtain

\[ \lim_{y \to 0} \frac{xy^4}{x^2 + y^8} = \lim_{x \to 0} \frac{0}{x^2} = 0 \]

and if we let \((x, y)\) approach \((0, 0)\) along the curve \(x = y^4\), then we obtain

\[ \lim_{(x,y) \to (0,0) \atop (x=y^4)} \frac{xy^4}{x^2 + y^8} = \lim_{y \to 0} \frac{y^4y^4}{(y^4)^2 + y^8} = \frac{1}{2}. \]

Since two different limits can be obtained using different paths of approach to \((0, 0)\), we conclude that the limit in question does not exist.

3. Level curves for a function, \(z = f(x, y)\), are shown below. Determine whether the indicated partial derivatives are positive, negative, or zero at the point \((x_0, y_0)\). (Circle the correct choices.)
4. Find an equation for the tangent plane to the surface

\[ \mathbf{r}(s, t) = s^2 \mathbf{i} + 2s \sin(t) \mathbf{j} + s \cos(t) \mathbf{k} \]

at the point on the surface corresponding to \((s, t) = (1, 0)\).

**Solution:**

\[ \mathbf{r}_s(s, t) = 2s \mathbf{i} + 2 \sin(t) \mathbf{j} + \cos(t) \mathbf{k} \]

\[ \mathbf{r}_t(s, t) = 2s \cos(t) \mathbf{j} - s \sin(t) \mathbf{k} \]

\[ \mathbf{r}_s(1, 0) = 2\mathbf{i} + \mathbf{k} \]

\[ \mathbf{r}_t(1, 0) = 2\mathbf{j} \]

\[ \mathbf{r}_s(1, 0) \times \mathbf{r}_t(1, 0) = -2\mathbf{i} + 4\mathbf{k} \]

so we can take \( \mathbf{n} = \mathbf{i} - 2\mathbf{k} \) as a normal vector.
Also, \((s, t) = (1, 0)\) corresponds to the point \((x_0, y_0, z_0) = (1, 0, 1)\) on the surface, so an equation of the tangent plane at this point is

\[
1 (x - 1) + 0 (y - 0) - 2 (z - 1) = 0
\]

or

\[
x - 1 - 2z + 2 = 0
\]

or

\[
x - 2z = -1.
\]

A picture of this surface and this tangent plane are shown below.

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5. For

\[
w = \ln \left( x^2 + y^2 + z^2 \right)
\]

\[
x = s^2
\]

\[
y = 2s - t
\]

\[
z = \sqrt{2s + t}
\]

use the Chain Rule to find \(\partial w/\partial t\). (No need to simplify your answer.)

**Solution:**

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
\]

\[
= \frac{2x}{x^2 + y^2 + z^2} (0) + \frac{2y}{x^2 + y^2 + z^2} (-1)
\]

\[
+ \frac{2z}{x^2 + y^2 + z^2} \left( \frac{1}{\sqrt{2s + t}} \right)
\]

6. For the function

\[
f(x, y) = \tan(x + 2y),
\]
(a) Find $\nabla f (0, 0)$.

(b) For the unit vector $\mathbf{u} = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, find $D_u f (0, 0)$.

**Solution:**

$$\nabla f (x, y) = f_x (x, y) \mathbf{i} + f_y (x, y) \mathbf{j}$$

$$= \sec^2 (x + 2y) \mathbf{i} + 2 \sec^2 (x + 2y) \mathbf{j}$$

so

$$\nabla f (0, 0) = \sec^2 (0) \mathbf{i} + 2 \sec^2 (0) \mathbf{j} = \mathbf{i} + 2 \mathbf{j}.$$ 

Also

$$D_u f (0, 0) = \nabla f (0, 0) \cdot \mathbf{u} = (1) \left( -\frac{1}{2} \right) + 2 \left( \frac{\sqrt{3}}{2} \right) = \sqrt{3} - 1.$$