MATH 2203 – Final Exam (Version 2) Solutions
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Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Match each of the two–dimensional vector fields, \( \mathbf{F} \), given in parts a–e with its graph (given in 1–10 on the separately attached sheet).

   a) \[ \mathbf{F}(x, y) = \frac{-x}{\sqrt{x^2 + y^2 + 1}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + 1}} \mathbf{j} \] matches Graph 8.

   b) \[ \mathbf{F}(x, y) = \frac{y}{\sqrt{x^2 + y^2 + 1}} \mathbf{i} + \frac{-x}{\sqrt{x^2 + y^2 + 1}} \mathbf{j} \] matches Graph 4.

   c) \[ \mathbf{F}(x, y) = \frac{-y}{\sqrt{x^2 + y^2 + 1}} \mathbf{i} + \frac{x}{\sqrt{x^2 + y^2 + 1}} \mathbf{j} \] matches Graph 9.

   d) \[ \mathbf{F}(x, y) = \frac{x}{\sqrt{x^4 + x^2 + 1}} \mathbf{i} + \frac{x^2}{\sqrt{x^4 + x^2 + 1}} \mathbf{j} \] matches Graph 10.

   e) \[ \mathbf{F}(x, y) = \frac{-y}{\sqrt{x^2 + y^2 + 1}} \mathbf{i} + \frac{-x}{\sqrt{x^2 + y^2 + 1}} \mathbf{j} \] matches Graph 1.

2. Let \( \mathbf{F} \) be the vector field

\[ \mathbf{F}(x, y) = \sin(x) \mathbf{i} + \cos(y) \mathbf{j} \] and let \( C \) be the oriented parabolic curve \( y = x^2 \) starting at the point \((-1, 1)\) and ending at the point \((2, 4)\). (The vector field and curve are pictured below.)
Evaluate the line integral
\[ \int_C \mathbf{F} \cdot d\mathbf{r}. \]

Solution: Observe that \( \mathbf{F} \) is a conservative vector field. This means that the value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \) does not depend on the path chosen in getting from the point \((-1, 1)\) to the point \((2, 4)\). A potential function for \( \mathbf{F} \) is \( f(x, y) = \sin(y) - \cos(x) \). Therefore

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(-1,1)}^{(2,4)} \mathbf{F} \cdot d\mathbf{r} \\
= f(2, 4) - f(-1, 1) \\
= (\sin(4) - \cos(2)) - (\sin(1) - \cos(-1)) \\
\approx -0.642.
\]

3. (a) Determine whether or not the vector field
\[ \mathbf{F}(x, y) = x \sin(y) \mathbf{i} + y \mathbf{j} \]
is conservative. If you determine that \( \mathbf{F} \) is not conservative, then stop here (do not proceed to parts b and c).

(b) Find a function \( f \) such that \( \mathbf{F} = \nabla f \) and proceed to part c.

(c) Use the Fundamental Theorem of Line Integrals to evaluate
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where \( C \) is any curve in \( \mathbb{R}^2 \) that begins at the point \((-1, 1)\) and ends at the point \((2, 4)\).

Solution: Observe that \( \partial Q / \partial x \neq \partial P / \partial y \). Therefore \( \mathbf{F} \) is not conservative.
4. Let the curves $C_1$, $C_2$, and $C_3$ be as follows:

$C_1$ is the line segment beginning at the point $(0, 1)$ and ending at the point $(0, 0)$.

$C_2$ is the line segment beginning at the point $(0, 0)$ and ending at the point $(1, 0)$.

$C_3$ is the curve $y = 1 - x^2$ beginning at the point $(1, 0)$ and ending at the point $(0, 1)$.

Also let $C = C_1 + C_2 + C_3$. (This curve is pictured below.)

Evaluate $\int_C x \, dx + y \, dy$ by two methods:

(a) directly

(b) using Green’s Theorem.

**Solution:** The Green’s Theorem way is easier! Since $P(x, y) = x$ and $Q(x, y) = y$, we have $\partial Q/\partial x - \partial P/\partial y = 0 - 0 = 0$ and Green’s Theorem gives us

$$\oint_C x \, dx + y \, dy = \iint_D (\partial Q/\partial x - \partial P/\partial y) \, dA = \iint_D (0) \, dA = 0.$$  

We can also evaluate this line integral directly as follows: $C_1$ can be parameterized as $x = 0, y = 1 - t, 0 \leq t \leq 1$ and we have

$$\int_{C_1} x \, dx + y \, dy = - \int_0^1 (1 - t) \, dt = -\frac{1}{2}$$
$C_2$ can be parameterized as $x = x, y = 0, 0 \leq x \leq 1$ and we have
\[ \int_{C_2} x \, dx + y \, dy = \int_0^1 x \, dx = \frac{1}{2} \]

$C_3$ can be parametrized as
\[
\begin{align*}
x &= 1 - t \\
y &= 1 - (1 - t)^2 = 2t - t^2 \\
0 &\leq t \leq 1
\end{align*}
\]
and we have
\[
\int_{C_3} x \, dx + y \, dy = \int_0^1 ((1 - t) (-1) + (2t - t^2) (2 - 2t)) \, dt = 0.
\]
This gives us
\[
\int_C x \, dx + y \, dy = \int_{C_1} x \, dx + y \, dy + \int_{C_2} x \, dx + y \, dy + \int_{C_3} x \, dx + y \, dy = -\frac{1}{2} + \frac{1}{2} + 0 = 0.
\]

5. A vector field, $F$, is said to be **irrotational** if $\text{curl}(F) = 0$ and is said to be **incompressible** is $\text{div}(F) = 0$.

(a) Explain why any vector field of the form
\[
F(x, y, z) = f(x) \mathbf{i} + g(y) \mathbf{j} + h(z) \mathbf{k}
\]
(where $f$, $g$, and $h$ are differentiable functions) is irrotational.

**Explanation:**
\[
\text{curl}(F) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f(x) & g(y) & h(z)
\end{vmatrix}
= \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} - \left( \frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) \mathbf{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k}
= (0 - 0) \mathbf{i} - (0 - 0) \mathbf{j} + (0 - 0) \mathbf{k}
= 0
\]
so $F$ is irrotational.

(b) Explain why any vector field of the form
\[
F(x, y, z) = f(y, z) \mathbf{i} + g(x, z) \mathbf{j} + h(x, y) \mathbf{k}
\]
(where $f$, $g$, and $h$ are differentiable functions) is incompressible.

**Explanation:**
\[
\text{div}(F) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0 + 0 + 0 = 0
\]
so $F$ is incompressible.
6. Let $S$ be the surface of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$ together with its top and bottom disks. This surface is pictured below. Evaluate

$$\iint_S (x^2 + y^2 + z^2) \, dS.$$ 

**Solution:** Let $S_0$ be the outward oriented bottom disk. Let $S_2$ be the outward oriented bottom disk. Let $S_c$ be the outward oriented cylinder. The surface $S_0$ is the graph of $z = 0$ so

$$\iint_{S_0} (x^2 + y^2 + z^2) \, dS = \iint_D (x^2 + y^2 + z^2) \sqrt{(zx)^2 + (zy)^2 + 1} \, dA$$

$$= \iint_D (x^2 + y^2 + (0)^2) \sqrt{(0)^2 + (0)^2 + 1} \, dA$$

$$= \int_0^{2\pi} \int_0^3 r^2 r \, dr \, d\theta = \frac{81\pi}{2}.$$
The surface $S_2$ is the graph of $z = 2$ so
\[
\iint_{S_2} \left( x^2 + y^2 + z^2 \right) \, dS = \iint_{D} \left( x^2 + y^2 + z^2 \right) \sqrt{(z_x)^2 + (z_y)^2 + 1} \, dA
\]
\[
= \iint_{D} \left( x^2 + y^2 + (2)^2 \right) \sqrt{(0)^2 + (0)^2 + 1} \, dA
\]
\[
= \int_{0}^{2\pi} \int_{0}^{3} (r^2 + 4) \, r \, dr \, d\theta = \frac{153\pi}{2}.
\]

The surface $S_c$ can be parameterized as
\[
\begin{align*}
x &= 3 \cos (\theta) \\
y &= 3 \sin (\theta) \\
z &= z \\
0 &\leq \theta \leq 2\pi \\
0 &\leq z \leq 2
\end{align*}
\]
and we see that
\[
\mathbf{r}_\theta \times \mathbf{r}_z = (-3 \sin (\theta) \mathbf{i} + 3 \cos (\theta) \mathbf{j}) \times \mathbf{k} = 3 \cos (\theta) \mathbf{i} + 3 \sin (\theta) \mathbf{j}
\]
which gives
\[
|\mathbf{r}_\theta \times \mathbf{r}_z| = 3.
\]
Thus
\[
\iint_{S_c} \left( x^2 + y^2 + z^2 \right) \, dS = \iint_{D} \left( 9 + z^2 \right) (3) \, dA
\]
\[
= 3 \int_{0}^{2} \int_{0}^{2\pi} (9 + z^2) \, d\theta \, dz
\]
\[
= 124\pi.
\]
Thus
\[
\iint_{S} \left( x^2 + y^2 + z^2 \right) \, dS = \iint_{S_0+S_2+S_c} \left( x^2 + y^2 + z^2 \right) \, dS
\]
\[
= \frac{81\pi}{2} + \frac{153\pi}{2} + 124\pi
\]
\[
= 241\pi.
\]