Let $C$ be the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$. Use Green's Theorem to evaluate

$$\int_C \left( y + e^{\sqrt{x}} \right) \, dx + \left( 2x + \cos (y^2) \right) \, dy$$

(where $C$ is assumed to have the positive orientation).

**Solution:** The curve $C = C_1 + C_2$ (and the region, $D$, that it bounds) is shown below. (We will treat $D$ as a Type I region.)
By Green’s Theorem,

\[
\int_C (y + e^{\sqrt{x}}) \, dx + (2x + \cos y^2) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[
= \iint_D (2 - 1) \, dA
\]

\[
= \int_0^1 \int_{x^2}^{\sqrt{x}} dy \, dx
\]

\[
= \int_0^1 (\sqrt{x} - x^2) \, dx
\]

\[
= \frac{2}{3} x^{3/2} \bigg|_{x=1}^{x=1} - \frac{1}{3} x^3 \bigg|_{x=0}^{x=0}
\]

\[
= \frac{1}{3}
\]