Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. You must write arrows over vectors! Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Show that
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 \cos^2(y)}{x^2 + 2y^2}
\]
does not exist by demonstrating that there are two different paths of approach to (0, 0) along which different limits are obtained.

Solution: By approaching (0, 0) along the x axis, we obtain
\[
\lim_{(x,y)\to(0,0), x \to 0} \frac{x^2 \cos^2(y)}{x^2 + 2y^2} = \lim_{x\to 0} \frac{x^2 \cos^2(0)}{x^2 + 2(0)^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.
\]

By approaching (0, 0) along the y axis, we obtain
\[
\lim_{(x,y)\to(0,0), y \to 0} \frac{x^2 \cos^2(y)}{x^2 + 2y^2} = \lim_{y\to 0} \frac{(0)^2 \cos^2(y)}{(0)^2 + 2y^2} = \lim_{y\to 0} 0 = 0.
\]

Therefore
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 \cos^2(y)}{x^2 + 2y^2}
\]
does not exist.

2. For the function
\[
u = xe^{-t} \sin(\theta),
\]
compute \(\partial u/\partial x\), \(\partial u/\partial t\), and \(\partial u/\partial \theta\). (This is a very short problem.)

Solution:
\[
\frac{\partial u}{\partial x} = e^{-t} \sin(\theta)
\]
\[
\frac{\partial u}{\partial t} = -xe^{-t} \sin(\theta)
\]
\[
\frac{\partial u}{\partial \theta} = xe^{-t} \cos(\theta).
\]
3. Find the linear approximation, \( L(x, y) \), for the function
\[
f(x, y) = \sqrt{20 - x^2 - 7y^2}
\]
at the point \((2, 0)\).

**Solution:** First note that \( f(2, 0) = 4 \). Also,
\[
\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{20 - x^2 - 7y^2}} = \frac{-x}{z} \\
\frac{\partial f}{\partial y} = \frac{-7y}{\sqrt{20 - x^2 - 7y^2}} = \frac{-7y}{z}
\]
and thus
\[
\frac{\partial f}{\partial x}(2, 0) = \frac{-2}{4} = -\frac{1}{2} \\
\frac{\partial f}{\partial y}(2, 0) = \frac{-7(0)}{4} = 0.
\]
The linear approximation is thus
\[
L(x, y) = 4 - \frac{1}{2} (x - 2) + 0 (y - 0)
\]
or
\[
L(x, y) = -\frac{1}{2} x + 5.
\]

4. Suppose that
\[
z = \sin(x) \tan(y)
\]
where
\[
x = 3s + t \\
y = s - t.
\]
Use the Chain Rule to find \( \partial z / \partial s \) and \( \partial z / \partial t \). (You do not need to simplify your answers.)

**Solution:** By the Chain Rule
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \cos(x) \tan(y) (3) + \sin(x) \sec^2(y) (1)
\]
and
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \cos(x) \tan(y) (1) + \sin(x) \sec^2(y) (-1)
\]

5. Find the directional derivative of the function \( f(x, y) = x^2e^y \) at the point \((2, 0)\) in the direction of the unit vector \( \mathbf{u} = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} \).
Solution: Since
\[
\frac{\partial f}{\partial x} = 2xe^y \\
\frac{\partial f}{\partial y} = x^2 e^y,
\]
we have
\[
\frac{\partial f}{\partial x} (2, 0) = 4 \\
\frac{\partial f}{\partial y} (2, 0) = 4
\]
and thus
\[
\nabla f (2, 0) = 4\mathbf{i} + 4\mathbf{j}.
\]
This gives
\[
D_u f (2, 0) = \nabla f (2, 0) \cdot \mathbf{u} = (4) \left( \frac{\sqrt{2}}{2} \right) + (4) \left( \frac{\sqrt{2}}{2} \right) = 4\sqrt{2}.
\]

6. A hill, whose equation is
\[
z = 1000 - 0.005x^2 - 0.01y^2
\]
(where \( x, y, \) and \( z \) are measured in meters) is pictured below. The positive \( x \) axis points to the east and the positive \( y \) axis points to the north. This means that the view of the hill shown below is from the northeast.

Suppose that you are standing on the hill at the point whose coordinates are \((60, 40, 966)\).

(a) If you walk due south, will you start to ascend or descend? At what rate?
(b) If you walk northwest, will you start to ascend or descend? At what rate?
(c) In which direction should you walk if you wish to ascend at the greatest possible rate? At what rate will you ascend if you walk in that direction?

Solution: Note that
\[
\frac{\partial f}{\partial x} = -0.01x \\
\frac{\partial f}{\partial y} = -0.02y
\]
and thus
\[
\nabla f (60, 40) = -0.01 (60) \mathbf{i} - 0.02 (40) \mathbf{j} = -0.6\mathbf{i} - 0.8\mathbf{j}.
\]
You should walk in the direction of this vector in order to get the steepest ascent. What is this direction? Solving
\[
\tan (\theta) = \frac{-0.8}{-0.6}
\]
and noting that \( \theta \) is a third quadrant angle, we obtain \( \theta \approx 233.13^\circ \). This means that the direction is W53.13°S. The rate of ascent in this direction is

\[
|\nabla f (60, 40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1 \text{ meter per meter.}
\]

We have just answered part c. The answer to part a is

\[
D_{-\mathbf{j}} f (60, 40) = \nabla f (60, 40) \cdot (-\mathbf{j}) = 0.8 \text{ meters per meter.}
\]

To answer part b, we note that the unit vector that points to the northwest is

\[
\mathbf{u} = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}.
\]

Hence, if you walk northwest, then

\[
D_{\mathbf{u}} f (60, 40) = \nabla f (60, 40) \cdot \mathbf{u}
\]

\[
= (-0.6) \left( -\frac{\sqrt{2}}{2} \right) + (-0.8) \left( \frac{\sqrt{2}}{2} \right)
\]

\[
\approx -0.14 \text{ meters per meter.}
\]

(You are descending if you walk to the northwest.)