Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. You must write arrows over vectors! Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. A table of values for a function, $f$, defined on the rectangle $R = [1, 3] \times [0, 4]$ is given below.

(a) Estimate

$$\iint_R f(x, y) \, dA$$

using the Midpoint Rule with $m = n = 2$.

(b) Estimate this same integral using $m = n = 4$ and choosing the sample points to be the points farthest from the origin. (It is essential to show your work in detail in order to receive credit.)

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</table>

Solution: First we will do part a. For $m = n = 2$, we have

$$\Delta x = \frac{3 - 1}{2} = 1$$
$$\Delta y = \frac{4 - 0}{2} = 2$$
$$\Delta A = \Delta x \cdot \Delta y = 2.$$ 

The subrectangles we obtain are thus $[1, 2] \times [0, 2]$, $[1, 2] \times [2, 4]$, $[2, 3] \times [0, 2]$, and $[2, 3] \times [2, 4]$. Their respective midpoints are $(1.5, 1)$, $(1.5, 3)$, $(2.5, 1)$ and $(2.5, 3)$. This gives us the following estimate of the given integral:

$$(f(1.5, 1) + f(1.5, 3) + f(2.5, 1) + f(2.5, 3)) \Delta A$$
$$= (0 + (-8) + 5 + (-1)) (2) = -8.$$
Now we do part b: For \( m = n = 4 \), we have
\[
\Delta x = \frac{3 - 1}{4} = \frac{1}{2}, \quad \Delta y = \frac{4 - 0}{4} = 1
\]
\[
\Delta A = \Delta x \cdot \Delta y = \frac{1}{2}.
\]
The subrectangles we obtain are thus
\[
[1, 1.5] \times [0, 1], \quad [1, 1.5] \times [1, 2], \quad [1, 1.5] \times [2, 3],
\]
\[
[1, 1.5] \times [3, 4], \quad [1.5, 2] \times [0, 1], \quad [1.5, 2] \times [1, 2], \quad [1.5, 2] \times [2, 3],
\]
\[
[1.5, 2] \times [3, 4], \quad [2, 2.5] \times [0, 1], \quad [2, 2.5] \times [1, 2], \quad [2, 2.5] \times [2, 3],
\]
\[
[2, 2.5] \times [3, 4], \quad [2.5, 3] \times [0, 1], \quad [2.5, 3] \times [1, 2], \quad [2.5, 3] \times [2, 3],
\]
\[
[2.5, 3] \times [3, 4].
\]
Using the points in these subrectangles that are farthest from the origin, we obtain the following estimate of the integral:
\[
(0 - 4 - 8 - 6 + 3 + 0 - 5 - 8 + 5 + 3 - 1 - 4 + 8 + 6 + 3 + 0) \left(\frac{1}{2}\right) = -4.
\]

2. Calculate (including all details) the iterated integral
\[
\int_0^1 \int_1^2 \frac{x e^x}{y} \, dy \, dx.
\]

**Solution:** This iterated integral is separable:
\[
\int_0^1 \int_1^2 \frac{x e^x}{y} \, dy \, dx = \left( \int_0^1 x e^x \, dx \right) \left( \int_1^2 \frac{1}{y} \, dy \right) = (1) (\ln (2)) = \ln (2)
\]

3. Use polar coordinates to evaluate the double integral
\[
\iint_D (2x - y) \, dA
\]

where \( D \) is the disk with center at the origin and with radius 2.

**Solution:**
\[
\iint_D (2x - y) \, dA = \int_0^{2\pi} \int_0^2 (2r \cos (\theta) - r \sin (\theta)) \, r \, dr \, d\theta = 0.
\]

4. Use rectangular coordinates to evaluate the double integral
\[
\iint_D (2x - y) \, dA
\]

where \( D \) is the disk with center at the origin and with radius 2.

**Solution:** We view \( D \) as a Type I region:
\[
\iint_D (2x - y) \, dA = \int_{-1}^{1} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x - y) \, dy \, dx = 0
\]
5. Find the center of mass of the lamina, $D$, that occupies the region bounded by the parabola $x = y^2$ and the line $y = x - 2$ with density function $\rho(x, y) = 3$.

**Solution:** We view $D$ as a Type I region. The mass of the lamina is

$$M = \iint_D 3\,dA = \int_{-1}^{2} \int_{y^2}^{y+2} 3\,dx\,dy = \frac{27}{2}.$$

The center of mass is at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{2}{27} \iint_D x\,dA = \frac{2}{27} \int_{-1}^{2} \int_{y^2}^{y+2} 3x\,dx\,dy = \frac{8}{5},$$

and

$$\bar{y} = \frac{2}{27} \iint_D y\,dA = \frac{2}{27} \int_{-1}^{2} \int_{y^2}^{y+2} 3y\,dx\,dy = \frac{1}{2}.$$